First Seminar on
Reliability Theory and its Applications
27-28, May 2015

On Mean Residual Life Ordering Among Weighted-k-out-of-n Systems

Rahmani, R. [1] , Izadi, M. [2] and Khaledi, B. [3]

Department of Statistics, Razi University

Abstract

Consider a system consisting of n binary components with different contributions (weights) on determining the state of the system. The system is known as weighted-k-out-of-n system when it works iff the total weight of working components are greater than a pre-specified value k. Suppose that this system has the property that, with probability 1, operates as long as at least $n-s+1$ components operate ($s \leq n$). In this paper, we compare two such systems with respect to their mean residual life function under the condition that $n-r+1$ components ($r \leq s$) of the systems are working at time t.

Keywords: Weighted-k-out-of-n system, Mean residual life, Usual stochastic order.

1 Introduction

Consider a system consisting of n binary components with different contributions on determining the state of the system. Let w_i, $i = 1, \ldots, n$, be the positive weight of the component i. The system is known as weighted-k-out-of-n system when it works iff the total weight of working components are greater than a pre-specified value k, that is, $\sum_{i=1}^{n} w_i X_i \geq k$ where X_i is the state of the component i, $i = 1, \ldots, n$. The weighted-k-out-of-n system was introduced by Wu and Chen (1994) and studied by many researchers including Higashiyama (2001), Chen and Yang (2005), Samaniego and Shaked (2008) and Eryilmaz and Bozbulut (2014).

[1] rahmani.stat@gmail.com
[2] m.izadi@razi.ac.ir
[3] bkhaledi@hotmail.com

One of the most important characteristics of a system in the reliability theory is the mean residual life (MRL) function. Let T be the random lifetime of a system with survival function $\bar{F}$. Then, the MRL function of the system at time t is given by

$$m(t) = E(T - t|T > t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} \quad , \quad t > 0.$$

We refer the reader to Kotz and Shanbhag (1980), Guess and Proschan (1988), Shaked and Shanthikumar (2007) and Asadi and Goliforushani (2008) for some results regarding the MRL function.

Now, consider a weighted-k-out-of-n system consisting of n components with lifetimes $T_1, \ldots, T_n$ and the weight vector $\mathbf{w} = (w_1, \ldots, w_n)$. Suppose that this system has the property that, with probability 1, operates as long as at least $n-s+1$ components operate $(s \leq n)$. We denote this sytem by s-weighted-k-out-of-n system. Under the condition that at time t at least $(n - r + 1)$ components $(r \leq s)$ are alive, the residual life of the system is

$$(T_{\mathbf{w}} - t|T_{(r)} > t), \quad r = 1, \ldots, s$$

where $T_{\mathbf{w}}$ is the lifetime of the system and $T_{(r)}$ is the rth order statistics of $T_1, \ldots, T_n$. The MRL function of the above system can be defined as

$$m_{\mathbf{w}}^{r,s}(t) = E[T_{\mathbf{w}} - t|T_{(r)} > t]. \tag{1}$$

In this paper, we are interested in the comparison of such weighted-k-out-of-n systems (described above) with respect to their mean residual life function defined in (1).

We end this section by recalling the signature vector of a system and the usual stochastic order that will be use later in the paper. Consider a system with lifetime T whose component lifetimes $T_1, \ldots, T_n$ are independent and identically distributed. Samaniego (1985) defined the signature vector of the system as a probability vector $\mathbf{q} = (q_1, \ldots, q_n)$ with

$$q_i = P\{T = T_{(i)}\}, \quad i = 1, \ldots, n.$$

Let X and Y be two random variables with survival function $\bar{F}$ and $\bar{G}$, respectively. X is said to be less than Y in the usual stochastic order (denoted by $X \leq_{st} Y$) if $\bar{F}(x) \leq \bar{G}(x)$ for all $x \in R$.

2 Main results

Let $T_{\mathbf{w}}$ and $T_{\mathbf{w}'}$ be the lifetime of two weighted-k-out-of-n systems with independent and identically distributed component lifetimes T_1, ..., T_n and $T'_1, \ldots, T'_n$, weight vectors $\mathbf{w} = (w_1, \ldots, w_n)$ and $\mathbf{w}' = (w'_1, \ldots, w'_n)$ and signature vectors $\mathbf{q} = (q_1, \ldots, q_n)$ and $\mathbf{q}' = (q'_1, \ldots, q'_n)$, respectively.

Our first result is in the following theorem.

Theorem 1. *Consider two weighted-k-out-of-n systems as above. If* $\mathbf{w} \leq \mathbf{w}'$*, i.e.* $w_i \leq w'_i, i = 1, \ldots, n$*, then* $\mathbf{q} \leq_{st} \mathbf{q}'$*.*

Proof. It is enough to show that $\sum_{i=j}^{n} q_i \leq \sum_{i=j}^{n} q'_i$, for $j = 1, \ldots, n$.

$$\begin{aligned}
\sum_{i=j}^{n} q_i &= P(T_{\mathbf{w}} \geq T_{(j)}) \\
&= \sum_{\{(i_1,\ldots,i_n);\sum_{h=j}^{n} w_{i_h} \geq k\}} P(T_{i_1} \leq \cdots \leq T_{i_n}) \\
&\leq \sum_{\{(i_1,\ldots,i_n);\sum_{h=j}^{n} w'_{i_h} \geq k\}} P(T_{i_1} \leq \cdots \leq T_{i_n}) \\
&= \sum_{\{(i_1,\ldots,i_n);\sum_{h=j}^{n} w'_{i_h} \geq k\}} P(T'_{i_1} \leq \cdots \leq T'_{i_n}) \\
&= P(T_{\mathbf{w}'} \geq T'_{(j)}) \\
&= \sum_{i=j}^{n} q'_i.
\end{aligned}$$

Now, consider an s-weighted-k-out-of-n system (introduced in Section 1) with independent and identically distributed component lifetimes $T_1, \ldots, T_n$. It is obvious that such a system has the signature vector of the form $\mathbf{q} = (0, \ldots, 0, q_s, q_{s+1}, \ldots, q_n)$. If $T_1, \ldots, T_n$ are distributed according to a common continuous distribution F, then

$$m_{\mathbf{w}}^{r,s}(t) = \sum_{i=r}^{n}\sum_{u=0}^{r-1}\sum_{v=u}^{i-1} q_i \binom{n}{u}\binom{n-u}{v-u} \frac{\int_t^\infty F^u(t)(F(x)-F(t))^{v-u}\bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1}\binom{n}{u}F^u(t)\bar{F}^{n-u}(t)} \tag{2}$$

Theorem 2. *Consider two s-weighted-k-out-of-n systems with weight vectors* $\mathbf{w}$ *and* $\mathbf{w}'$*, signature vectors* $\mathbf{q}$ *and* $\mathbf{q}'$*, both based on components with independent and identical lifetimes with common distribution F. Let* $m_{\mathbf{w}}^{r,s}(t)$ *and* $m_{\mathbf{w}'}^{r,s}(t)$ *be their respective mean residual life functions defined in (1). If* $\mathbf{q} \leq_{st} \mathbf{q}'$*, then* $m_{\mathbf{w}}^{r,s}(t) \leq m_{\mathbf{w}'}^{r,s}(t)$*.*

Proof. By interchanging the order of the summations in (2), we have that,

$$\begin{aligned}
m_{\mathbf{w}}^{r,s}(t) = &\sum_{u=0}^{r-1}\sum_{v=u}^{r-1}\Big(\sum_{i=r}^{n} q_i\Big)\binom{n}{u}\binom{n-u}{v-u}\frac{\int_t^\infty F^u(t)(F(x)-F(t))^{v-u}\bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1}\binom{n}{u}F^u(t)\bar{F}^{n-u}(t)} \\
&+\sum_{u=0}^{r-1}\sum_{v=r}^{n-1}\Big(\sum_{i=v+1}^{n} q_i\Big)\binom{n}{u}\binom{n-u}{v-u}\frac{\int_t^\infty F^u(t)(F(x)-F(t))^{v-u}\bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1}\binom{n}{u}F^u(t)\bar{F}^{n-u}(t)} \\
&\leq\sum_{u=0}^{r-1}\sum_{v=u}^{r-1}\Big(\sum_{i=r}^{n} q'_i\Big)\binom{n}{u}\binom{n-u}{v-u}\frac{\int_t^\infty F^u(t)(F(x)-F(t))^{v-u}\bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1}\binom{n}{u}F^u(t)\bar{F}^{n-u}(t)} \\
&+\sum_{u=0}^{r-1}\sum_{v=r}^{n-1}\Big(\sum_{i=v+1}^{n} q'_i\Big)\binom{n}{u}\binom{n-u}{v-u}\frac{\int_t^\infty F^u(t)(F(x)-F(t))^{v-u}\bar{F}^{n-v}(x)dx}{\sum_{u=0}^{r-1}\binom{n}{u}F^u(t)\bar{F}^{n-u}(t)} \\
= m_{\mathbf{w}'}^{r,s}(t).
\end{aligned}$$

The inequality follows from the assumption $\mathbf{q} \leq_{st} \mathbf{q}'$.

The following corollary follows from Theorems 1 and 2.
Corollary 1. Consider two s-weighted-k-out-of-n systems given in Theorem 2. If $\mathbf{w} \leq \mathbf{w}'$, then $m_{\mathbf{w}}^{r,s}(t) \leq m_{\mathbf{w}'}^{r,s}(t)$.

References

[1] Asadi, M. and Goliforushani, S. (2008), On the mean residual life function of coherent systems. *IEEE Transactions on Reliability*, **57**, 574–580.

[2] Chen, Y. and Yang, Q. (2005), Reliability of two-stage weighted k-out-of-n systems with components in common. *IEEE Transactions on Reliability*, **54**, 431–440.

[3] Eryilmaz, S. and Bozbulut, A.R. (2014), Computing marginal and joint Birnbaum, and Barlow-Proschan importances in weighted-k-out-of-n:G systems. *Computers & Industrial Engineering*, **72**, 255–260.

[4] Guess, F. and Proschan, F. (1988), Mean residual life, theory and applications, In: Krishnaiah, P.R., Rao, C.R.

[5] Higashiyama, Y. (2001), A factored reliability formula for weighted-k-out-of-n System. *Asia-Pacific Journal of Operational Research*, **18**, 61–66.

[6] Kotz, S. and Shanbhag, D.N. (1980), Some new approaches to probability distributions. *Advanced Applied Probability*, **12**, 903–921.

[7] Samaniego, F.J. (1985), On the closure of the IFR class under formation of coherent systems. *IEEE Transactions on Reliability*, **34**, 69-72.

[8] Samaniego, F.J. and Shaked, M. (2008), Systems with weighted components. *Statistics and Probability Letters*, **78**, 815–823.

[9] Shaked, M. and Shanthikumar, J.G. (1994), Stochastic Orders and their applications. Academic Press, Boston.

[10] Wu, J.S. and Chen, R.J. (1994), An algorithm for computing the reliability of a weighted-k-out-of-n system. *IEEE Transactions on Reliability*, **43**, 327–328.