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On Properties of Progressively Type-II Censored Conditionally N-Ordered Statistics Arising from a Non-Identical and Dependent Random Vector

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Abstract

In this paper, we investigate progressively Type-II censored conditionally N-ordered statistics arising from a system with identical as well as non-identical but dependent components, jointly distributed according to an Archimedean copula with completely monotone generator (PCCOSDNARCM-N). Our results generalized the results in Bairamov (2006) and is more flexible than those in practice, because of considering the dependency between components that is a common fact for real data.

Keywords: Archimedean copula, Order statistics, Progressive censoring, Progressively Type-II censored order statistics, Reliability systems.

1 Introduction

An experimenter may wish to reduce the size of a life test after having gained often critical early knowledge, while still obtaining information on later failures. The items removed make space for other experiments and reduce costs. Since in the real life we are face with dependent and non-identical data, we consider progressively Type-II right censored order statistics (PCOS-II) from a vector with a copula as the joint distribution function. In this case the marginal distributions are arbitrary and we are free to consider any desirable univariate distribution. Therefore, the marginal distribution of PCOS-II order statistics arising from dependent and non-identical sample according to copulas are applicable in real life. When we have a parallel system, the lifetime of the system equals the maximum of the lifetimes of the components. If we just record the lifetime of the system we ignore the information due to the components which can be used to obtain a

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more precise lifetime analysis. Hence a progressively Type-II right censored conditionally N-ordered statistics (PCCOS-N) sampling scheme is recommended. In this paper, we shall consider the PCCOS-N sampling scheme for systems that are neither identical nor independent. For example a common shock may affect the efficiency of the components or some stressful environment may lead dependency between components. We shall consider the dependency between systems using Archimedean copulas with completely monotone generators. These copulas have many desirable properties, for example,

- The class of Archimedean copulas with completely monotone generators contain several other such known copulas, see for example P.375 and 376 and 377 Joe [3].
- This family of copulas is MTP2, i.e. it has positive dependence property, (see e. g. [4] Muller and Scarsini, 2005), which is a suitable for lifetime data.
- Some goodness of fit tests exists for Archimedean copulas. In fact, we can consider which Archimedean copulas with completely monotone generator is best fit to a data set. (see e.g. [2]).

Recall that a function $\psi : \mathbb{R}_+ \to [0,1]$ is said to be *d-monotone* if $(-1)^k\psi^{(k)} \geq 0$ for $k \in \{1,\dots,d-2\}$ and $(-1)^{d-2}\psi^{(d-2)}$ is a decreasing and convex function, where $\psi^{(k)}$, the k-th derivative of the function ψ, exists for $k = 1,2,\dots,d-2$. If a function is d-monotone for all $d \in \mathbb{N}$, then it is said to be *completely monotone*. If a copula C_ψ has the form

$$C_\psi(u_1,\dots,u_n) = \psi\left(\sum_{i=1}^{n}\psi^{-1}(u_j)\right), \tag{1}$$

where $\psi : \mathbb{R}_+ \to [0,1]$ is an n-monotone $(n \geq 2)$ function such that $\psi(0) = 1$ and $\lim_{x\to\infty}\psi(x) = 0$, it is called an Archimedean copula with generator function ψ (see [3, 6, 5]). In this work, we concentrate on Archimedean copulas with completely monotone generator function. This family of copulas have applications in reliability theory in [7, 8]. Let $G(u) = \exp\{-\psi^{-1}(u)\}$, $u \in [0,1]$, and M_ψ be a distribution function with Laplace transform ψ. Then, an equivalent representation for (1) is given by

$$C_\psi(u_1,\dots,u_n) = \int_0^\infty \prod_{i=1}^{n} G^\alpha(u_i)dM_\psi(\alpha). \tag{2}$$

This representation is the key to the ensuing developments. Furthermore, we assume that ψ is strictly increasing and its inverse function ψ^{-1} is differentiable. Now, let us consider the PCCOS-N arising from either independent or dependent random vectors $\mathbf{X}_1,\dots,\mathbf{X}_n$. We assume that for $i = 1,\dots,N$, $\mathbf{X}_i = (X_i^1, X_i^2,\dots,X_i^n)$ are absolutely continuous, iid random vectors and let $T(\mathbf{X}_i)$ be the life time of the vector $\mathbf{X}^{\mathbf{R}}_{1:m:N}$. Then $T(\cdot)$ is a measurable $\mathbb{R}^p \setminus \mathbb{R}$ function. Under the PCCOS-N sampling scheme, $\mathbf{X}_1,\dots,\mathbf{X}_N$ are place on a life test. The first system to fail will be denoted by $\mathbf{X}^{\mathbf{R}}_{1:m:N}$. We have $T(\mathbf{X}^{\mathbf{R}}_{1:m:N}) = \min(T(\mathbf{X}_1),\dots,T(\mathbf{X}_N))$. We now remove R_1 system at random from the surviving set $\{\mathbf{X}_1,\dots,\mathbf{X}_N\} \setminus \mathbf{X}^{\mathbf{R}}_{1:m:N}$. After the ith failure, occurring at $N(\mathbf{X}^{\mathbf{R}}_{i:m:N})$, R_i surviving systems are removed at random. The procedure terminates at the mth step, where $R_1 + \cdots + R_m + m = N$. Bairamov [1] showed that the joint pdf of the first r PCCOS-N, $r = 1,2,\dots,m$, can be represented as follows

$$\begin{aligned} &f_{\mathbf{X}^{\mathbf{R}}_{1:m:N},\dots,\mathbf{X}^{\mathbf{R}}_{r:m:N}}(\mathbf{x}_1,\dots,\mathbf{x}_r) \\ &\quad= \Big(\prod_{j=1}^{r}\gamma_j\Big)f(\mathbf{x}_r)\Big(\bar{H}(\mathbf{x}_r)\Big)^{\gamma_r-1}\prod_{j=1}^{r-1}f(\mathbf{x}_j)\Big(\bar{H}(\mathbf{x}_j)\Big)^{R_j}, \end{aligned} \tag{3}$$

where $\gamma_j = n - \sum_{v=1}^{j-1}(R_v+1) = \sum_{v=j}^{m}(R_v+1)$, $1 \leq v \leq m$, $\gamma_1 = n$, and $H(\mathbf{x}) = P\{T(\mathbf{X}) \leq T(\mathbf{x})\}$. Here, we consider progressively Type-II censored conditionally N-ordered statistics arising from a system with identical as well as non-identical but dependent components, jointly distributed according to an Archimedean copula with completely monotone generator
(PCCOSDNARCM-N).

2 Main results

In this section, we obtain the joint and marginal density function of PCCOSDNARCH-N arising from a random vector $\mathbf{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_N)$ with joint survival function

$$P(\mathbf{X} > \mathbf{x}) = \int_0^\infty \prod_{i=1}^{N} G^\alpha(\bar{F}_i(\mathbf{x}_i))dM_\psi(\alpha)\,, \tag{4}$$

where $\bar{F}_i(\mathbf{x}) = P(\mathbf{X}_i > \mathbf{x})$ is the joint survival function of $\mathbf{X}_i$, $i = 1, \ldots, n$. For this purpose, let $\bar{H}_i(\mathbf{x}) = P\{T(\mathbf{X}_i) > T(\mathbf{x})\}$, then for $k = 1, \ldots, n-1$ the following identity is obvious.

$$\begin{aligned} &P(T(\mathbf{X}_1) > T(\mathbf{x}_1), \ldots, T(\mathbf{X}_k) > T(\mathbf{x}_k), \mathbf{X}_{k+1} > \mathbf{x}_{k+1}, \ldots, \mathbf{X}_n > \mathbf{x}_n) \\ &\quad = \psi\left(\sum_{i=1}^{k}\psi^{-1}(\bar{H}_i(\mathbf{x}_i)) + \sum_{i=k+1}^{n}\psi^{-1}(\bar{F}_i(\mathbf{x}_i))\right). \end{aligned} \tag{5}$$

We can obtain the joint density function of $\mathbf{X}^{\mathbf{R}}_{1:m:N}, \ldots, \mathbf{X}^{\mathbf{R}}_{m:m:N}$ using the following theorem.

Theorem 1. *For $n \in \mathbb{N}$, let S_n be the set of all permutations π of $(1, 2, \ldots, N)$. For brevity, let $\rho_r = R_1 + \cdots + R_r, 1 \leq r \leq m$, with $\rho_0 = 0$ and $\rho_m = N - m$. Then, the joint density of $\mathbf{X}^{\mathbf{R}}_{1:m:N}, \ldots, \mathbf{X}^{\mathbf{R}}_{m:m:N}$ is given by*

$$\begin{aligned} f_{\mathbf{X}^{\mathbf{R}}}(\mathbf{t}_1, \ldots, \mathbf{t}_m) \quad &= \quad \int_0^\infty \frac{1}{(N-1)!}\Big(\prod_{j=2}^{m}\gamma_j\Big)\sum_{\pi\in S_n}\prod_{j=1}^{m} f_{\pi(j)}(\mathbf{t}_j, \alpha) \\ &\quad \times\Big(\prod_{r=m+\rho_{j-1}+1}^{m+\rho_j}\bar{G}_{\pi(r)}(\mathbf{t}_j, \alpha)\Big)dM_\psi(\alpha)\,, \end{aligned} \tag{6}$$

where $T(\mathbf{t}_1) \leq \cdots \leq T(\mathbf{t}_m)$ and $\pi(i)$ is the i-th component of the permutation vector $\pi \in S_n, 1 \leq i \leq N$ and $\gamma_j = \sum_{i=j}^{m}(R_i + 1)$.

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