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Characterization of Bivariate Distribution by Mean Residual Life and Quantile Residual Life

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Abstract

Nair and Nair (1989) showed that bivariate mean residual life function characterizes the distribution uniquely. The subject of this paper is to verify how closely the bivariate quantile residual life function determines the distribution. It has been shown that like univariate case, two suitable bivariate quantile residual life can characterize the underlying distribution uniquely.

Keywords: Bivariate distribution function, Bivariate α-quantile residual life, Bivariate mean residual life

1 Introduction

When we deal with dependent components, extending reliability concepts to bivariate and multivariate seems inevitable. Same shocks on the components or excessive load survivors bear after their partners fail may cause their dependency. Many authors have introduced and studied bivariate or multivariate reliability concepts, e.g., Basu (1971) and Johnson and Kotz (1973) considered different versions of multivariate failure rate functions, Nair and Nair (1989) studied the mean residual life (MRL) concept for two possibly dependent components, and Roy (1994) studied multivariate aging classes and derived the chain of implications between them.

It is well-known that the MRL function determines the distribution function uniquely in the univariate case. Nair and Nair (1989) proved that the BMRL function uniquely determines the distribution function. Gupta and Longford (1984) determined the class of all distribution function F with the α-quantile residual life (α-QRL) function $q_\alpha(t)$. Song and Cho (1995) showed that in the class of continuous and strictly increasing distributions, two α-QRL functions $q_\alpha(t)$ and $q_\beta(t)$ which $\frac{\ln \bar{\alpha}}{\ln \beta}$ is irrational characterize the distribution function uniquely. Lin (2009) proved the result in the broader class of continuous models.

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2 The bivariate mean residual life

Assume the non-negative random vector $\mathbf{X} = (X_1, X_2)$ be lifetimes of two possibly dependent components. Let $\mathbf{X}$ follows absolutely continuous distribution F in the first quadrant of R^2, $Q = \{(x_1, x_2); x_i \geq 0\}$. Briefly, let $\mathbf{x} = (x_1, x_2)$ and $\mathbf{X} \geq \mathbf{x}$ stand for $X_i \geq x_i$, $i = 1, 2$. The well-known reliability function is $R(\mathbf{x}) = P(\mathbf{X} \geq \mathbf{x})$. The partial conditional reliability function for the i^{th} component is

$$R^{(i)}(x; \mathbf{x}) = P(X_i - x_i > x | \mathbf{X} \geq \mathbf{x}), \quad \mathbf{x} > \mathbf{0},\ x > 0,\ i = 1, 2. \tag{1}$$

The BMRL function can be written as

$$\mathbf{m}(\mathbf{x}) = E(\mathbf{X} - \mathbf{x} | \mathbf{X} > \mathbf{x}) = (m_1(\mathbf{x}), m_2(\mathbf{x})). \tag{2}$$

It can be verified that $m_1(\mathbf{x})R(\mathbf{x}) = \int_{x_1}^{\infty} R(t, x_2)dt$, and with differentiation

$$-\frac{\partial \ln R(\mathbf{x})}{\partial x_1} = (1 + \frac{\partial m_1(\mathbf{x})}{x_1})\frac{1}{m_1(\mathbf{x})}. \tag{3}$$

Similar equation holds in x_2 direction, and in turn by the fundamental theorem of line integrals, the integral $\int_{\mathbf{a}}^{\mathbf{b}} - \ln R(\mathbf{x})d\mathbf{x}$ is independent of the path joining $\mathbf{a}$ to $\mathbf{b}$ and

$$\int_{\mathbf{a}}^{\mathbf{b}} \nabla(-\ln R(\mathbf{t})) \cdot d\mathbf{t} = \ln R(\mathbf{a}) - \ln R(\mathbf{b}),$$

in which $\nabla(-\ln R(\mathbf{t})) = (-\frac{\partial \ln R(\mathbf{x})}{\partial x_1}, -\frac{\partial \ln R(\mathbf{x})}{\partial x_2})$. More specifically,

$$\int_{\mathbf{0}}^{\mathbf{x}} \nabla(-\ln R(\mathbf{t})) \cdot d\mathbf{t} = \ln R(\mathbf{0}) - \ln R(\mathbf{x}) = -\ln R(\mathbf{x}).$$

Nair and Nair (1989) considered the particular paths $(0, 0)$ to $(x_1, 0)$ and $(x_1, 0)$ to (x_1, x_2) to evaluate the left hand side of this equation, and obtained

$$R(\mathbf{x}) = \frac{m_1(0,0)m_2(x_1,0)}{m_1(x_1,0)m_2(x_1,x_2)} \exp\Big\{ -\int_0^{x_1} \frac{dt_1}{m_1(t_1,0)} - \int_0^{x_2} \frac{dt_2}{m_2(x_1,t_2)} \Big\}.$$

3 The bivariate quantile residual life

The i^{th} partial α-QRL function can be defined by

$$q_\alpha^{(i)}(\mathbf{x}) = \inf\{t_i : R^{(i)}(t_i; \mathbf{x}) = \bar{\alpha}\}, \quad \mathbf{x} \in \mathbb{R}^{+2}, \tag{4}$$

Taking $i = 1$, it simplifies to

$$q_\alpha^{(1)}(\mathbf{x}) = R_1^{-1}(\bar{\alpha}R(x_1, x_2); x_2) - x_1, \quad \mathbf{x} \in \mathbb{R}^{+2},$$

in which $R_1^{-1}(p; x_2) = \inf\{x_1 : \ R(x_1, x_2) = p\}$. Similarly

$$q_\alpha^{(2)}(\mathbf{x}) = R_2^{-1}(\bar{\alpha}R(x_1, x_2); x_1) - x_2.$$

The α-BQRL function can be constructed by gathering these two partial functions in a vector as $\mathbf{q}_\alpha(\mathbf{x}) = (q_\alpha^{(1)}(\mathbf{x}), q_\alpha^{(2)}(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^{+2}$. Johnson and Kotz (1973) considered the bivariate failure rate function $\mathbf{r}(\mathbf{x}) = (r_1(\mathbf{x}), r_2(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^2$ in which $r_i(\mathbf{x}) = -\frac{\partial}{\partial x_i} \ln R(\mathbf{x})$ represents the i^{th} partial failure rate function. As univariate case,

$$\int_{x_1}^{x_1+q_\alpha^{(1)}(\mathbf{x})} r_1(t, x_2)dt = -\ln \bar{\alpha}, \tag{5}$$

and

$$\int_{x_2}^{x_2+q_\alpha^{(2)}(\mathbf{x})} r_2(x_1, t)dt = -\ln \bar{\alpha}, \tag{6}$$

and consequently we have

$$1 + \frac{\partial}{\partial x_1} q_\alpha^{(1)}(\mathbf{x}) = \frac{r_1(x_1, x_2)}{r_1(x_1 + q_\alpha^{(1)}(x_1, x_2), x_2)}, \tag{7}$$

and

$$1 + \frac{\partial}{\partial x_2} q_\alpha^{(2)}(\mathbf{x}) = \frac{r_2(x_1, x_2)}{r_2(x_1, x_2 + q_\alpha^{(2)}(x_1, x_2))}, \tag{8}$$

respectively. As a result, $\frac{\partial}{\partial x_i} q_\alpha^{(i)}(\mathbf{x}) \geq -1$ for $i = 1, 2$. If $r_i(\mathbf{x})$ be increasing (decreasing) in x_i, then $q_\alpha^{(i)}(\mathbf{x})$ decreases (increases) in x_i.

4 Characterization by bivariate quantile residual life

Here, we are interested to investigate how closely the α-BQRL determines the distribution function. This leads us to solve the system of functional equations

$$\begin{cases} R(\varphi_1(\mathbf{x}), x_2) = \bar{\alpha} R(\mathbf{x}), \\ R(x_1, \varphi_2(\mathbf{x})) = \bar{\alpha} R(\mathbf{x}), \end{cases} \tag{9}$$

where $\varphi_i(\mathbf{x}) = x_i + q_\alpha^{(i)}(\mathbf{x})$, $i = 1, 2$.

It can be shown that every solution R of (9) can be written as

$$R(\mathbf{x}) = R_0(\mathbf{x}) h^*(\frac{\ln R_0(\mathbf{x})}{\ln \bar{\alpha}} - 1), \quad \mathbf{x} \in \mathbb{R}^{+2}, \tag{10}$$

or equivalently

$$R(\mathbf{x}) = R_0(\mathbf{x}) K(-\ln R_0(\mathbf{x})), \quad \mathbf{x} \in \mathbb{R}^{+2}, \tag{11}$$

in which R_0 is one special solution of (9) and K is a periodic function with period $-\ln \bar{\alpha}$. Clearly, $K(0) = 1$ and it must be restricted so that R be a reliability function.

Theorem 1. *Two BQRL functions $\mathbf{q}_\alpha(\mathbf{x})$ and $\mathbf{q}_\beta(\mathbf{x})$ which $\frac{\ln \bar{\alpha}}{\ln \bar{\beta}}$ is irrational uniquely determine the underlying distribution, namely F, in the class of continuous and strictly increasing bivariate distributions.*

Example 1. *It is simple to verify that the reliability function*

$$R_0(\mathbf{x}) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2\}, \quad \lambda_1, \lambda_2 > 0, \mathbf{x} \in \mathbb{R}^{+2}, \tag{12}$$

accommodate global constant BQRL function. Thus, by (11)*, the class of reliability functions*

$$R(\mathbf{x}) = \exp\{-\lambda_1 x_1 - \lambda_2 x_2\}(1 + \epsilon \sin(a\lambda_1 x_1 + a\lambda_2 x_2)), \quad \mathbf{x} \in \mathbb{R}^{+2}, \ |\epsilon| < \frac{1}{\sqrt{2}}, \ a > 0,$$

have global constant BQRL function for $\alpha = 1 - e^{-\frac{2k\pi}{a}}$, $k = 1, 2, \ldots$.

References

[1] Basu, A. P. (1971). Bivariate failure rate. *Journal of the American Statistical Association*, **66**, 103-104.

[2] Gupta, R. C. and Langford, E. S. (1984). On the determination of a distribution by its median residual life function: A functional equation. *Journal of Applied Probability,* **21**, 120-128.

[3] Johnson, N. L. and Kotz, S. (1973). A vector valued multivariate hazard rate. Bulletin of the International Statistical Institute, 45 (Book 1), 570-574.

[4] Lin, G. D. (2009). On the characterization of life distributions via percentile residual lifetimes. *Sankhya* Ser. A, **71**, Part 1, 64-72.

[5] Nair, K. R. and Nair, N. U. (1989). Bivariate mean residual life. *IEEE Transactions on Reliability*, **38**, 362-364.

[6] Roy, D. (1994). Classification of life distributions in multivariate models. *IEEE Transactions on Reliability*, **43(2)**, 224-229.

[7] Song, J. K. and Cho, G. Y. (1995). A note on percentile residual life. *Sankhya,* Ser. A, **57**, 333-335.