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Recent Advances in Comparisons of Coherent Systems Based on Inactivity Times

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Abstract

The purpose of the talk is to study the inactivity time of failed components of a coherent system consisting of n identical components with statistically independent lifetimes. Different aging and stochastic properties of this conditional random variable are obtained. Also we investigate stochastic properties of the inactivity time in the case where the component lifetimes are dependent random variables. Some results are extended to the case where the system has an arbitrary coherent structure with exchangeable components.

Keywords: Exchangeability, Joint reliability function, Signature, Likelihood ratio order.

1 Introduction

In the study of the reliability of engineering systems, the k-out-of-n structure plays a key role. A system with n components has a k-out-of-n structure if it operates as long as at least k of its components operate. The class of k-out-of-n systems is a special case of a class of systems which is known in the literature as coherent systems. A structure consisting of n components is known as a coherent system if the structure function of the system is monotone in its components, and each component of the system is relevant; see [2]. The concept of the signature of a coherent system, introduced by Samaniego [6], has become quite useful in studying the properties of coherent systems, and in comparing different systems. For a coherent system with lifetime T whose components' lifetimes $X_1, X_2, ..., X_n$ are statistically independent and identically distributed (i.i.d.) random variables with continuous distribution function F, the signature vector of the system is

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defined as a probability vector $\boldsymbol{s} = (s_1, s_2, ..., s_n)$ with $s_i = P\{T = X_{i:n}\}$, $i = 1, 2, ..., n$, where $X_{i:n}$ is the ith order statistic among $X_1, X_2, ..., X_n$; see [5], [6], [7].

Let X denote the lifetime of an alive unit having distribution F. Assuming that the unit has failed at or before time t, the inactivity time (IT) of X is defined as the conditional random variable $(t - X \mid X \leq t)$, which, in this context, represents the time that has elapsed since the failure of the unit. Among the researchers who have extended this concept to the coherent system, we can refer to [1], [4], [9].

On the basis of the structure of the coherent system, if the failure times of the components are not monitored continuously, then the exact failure times of some components of the system are unknown. Hence it might be important for reliability engineers and system designers to have some information about the time that has elapsed from a failure in the system. Suppose that an $(n - k + 1)$-out-of-n system is equipped with a warning light that comes up at the time of the failure of the jth component, $j < k$. The system is still working then, but the operator may now consider some maintenance or replacement policies. In this paper, we first study the time that has elapsed from the ith failure in the system, $i = 1, 2, ..., j$, given that the component with lifetime $X_{j:n}$ has failed at or before time t, but the system is working at time t; that is, the random variable

$$(t - X_{i:n} \mid X_{j:n} \leq t < X_{k:n}), \text{ for } i = 1, 2, ..., j, \text{ and } j < k.$$

This random variable is called the conditional IT of the component with lifetime $X_{i:n}$. Now, assume that a *coherent* system (with lifetime T) is alive at time t, and at least j components have failed by time t. We then define the conditional IT of the failed component with lifetime $X_{i:n}$ as $(t - X_{i:n} \mid X_{j:n} \leq t < T)$. In what follows, we investigate several interesting properties of the IT of $X_{i:n}$ for both $(n - k + 1)$-out-of-n and coherent systems.

We also investigate the properties of inactivity time of the components of a $(n-k+1)$-out-of-n system in the case where the components of the system are dependent. Let the vector $\boldsymbol{X} = (X_1, X_2, ..., X_n)$ denote the lifetimes of the components and assume that $\boldsymbol{X}$ has an arbitrary joint distribution function $F(t_1, t_2, ..., t_n)$. Assume that the system has failed at or before time t. Following the notation in [9], we define the inactivity time of the component with lifetime $X_{r:n}$, $r = 1, 2, ..., k$, at the system level as $(t - X_{r:n} \mid X_{k:n} \leq t)$.

2 Main results

Consider two $(n-k+1)$-out-of-n systems $\mathcal{S}_1$, and $\mathcal{S}_2$ with i.i.d. components $X_1, X_2, ..., X_n$, and $Y_1, Y_2, ..., Y_n$, respectively. The following result shows that, when the components of two systems are ordered in terms of reversed hazard rates, then the corresponding systems are stochastically ordered in terms of their IT [11]. For definitions of different stochastic orders, see [8].

Theorem 1. *Let $X_1 \leq_{\mathrm{rhr}} Y_1$. Then for any $t \geq 0$, and $1 \leq i \leq j < k \leq n$,*

$$(t - Y_{i:n} | Y_{j:n} \leq t < Y_{k:n}) \leq_{\mathrm{st}} (t - X_{i:n} | X_{j:n} \leq t < X_{k:n}).$$

It can be shown that the condition about the *rhr*-order in Theorem 1 cannot be replaced by similar properties on *hr*-order.

Theorem 2. *Let $X_1 \leq_{\mathrm{lr}} Y_1$. Then for any $t \geq 0$, $1 \leq i \leq j < k \leq n$, and $1 \leq i \leq p < q \leq m$,*

$$(t - Y_{i:n} \mid Y_{j:n} \leq t < Y_{k:n}) \leq_{\mathrm{lr}} (t - X_{i:m} \mid X_{p:m} \leq t < X_{q:m}),$$

whenever $n \leq m$, $j \leq p$, *and* $k \leq q$.

Let T be the lifetime of a coherent system with n i.i.d. components and signature vector $\boldsymbol{s} = (s_1, s_2, ..., s_n)$, and let $X_1, X_2, ..., X_n$ be the lifetimes of the components with a common absolutely continuous distribution F. We now present a result regarding the likelihood ratio ordering of the IT $(t - X_{i:n} \mid X_{j:n} \leq t < T)$ with respect to j.

Theorem 3. *If the distribution function F is absolutely continuous, then for $1 \leq i \leq j < n$, we have*

$$(t - X_{i:n} \mid X_{j:n} \leq t < T) \leq_{\mathrm{lr}} (t - X_{i:n} \mid X_{j+1:n} \leq t < T).$$

In the next theorem, we examine the implication of likelihood ratio and hazard rate orderings of the signature vectors of two systems.

Theorem 4. *Let T_1 and T_2 be the lifetimes of two coherent systems with common i.i.d. components $X_1, X_2, ..., X_n$, and signature vectors $\boldsymbol{s}^{(1)}$ and $\boldsymbol{s}^{(2)}$, respectively. If $\boldsymbol{s}^{(1)} \leq_{\mathrm{lr}} (\leq_{\mathrm{hr}})\boldsymbol{s}^{(2)}$, then for any $t \geq 0$,*

$$(t - X_{i:n} \mid X_{j:n} \leq t < T_1) \leq_{\mathrm{lr}} (\leq_{\mathrm{hr}})(t - X_{i:n} \mid X_{j:n} \leq t < T_2).$$

The reversed hazard rate function is an important measure in the study of engineering systems. Let X be an absolutely continuous random variable with the distribution function $F(t)$, and the probability density function $f(t)$. The *reversed hazard rate* function of X is defined as $r(t) = f(t)/F(t)$, for all t such that $F(t) > 0$. We say that X has a decreasing reversed hazard rate (DRHR) distribution if $r(t)$ is a decreasing function; for more details, see [3], [8]. In [11], it is shown that, when the component lifetimes of the system are DRHR, then the IT $(t - X_{i:n} \mid X_{j:n} \leq t < X_{k:n})$ is stochastically increasing in t.

Now consider a $(n-k+1)$-out-of-n system consisting of n components and assume that the components of the system are dependent with lifetimes $X_1, X_2, \ldots, X_n$.

Theorem 5. *If the density function of the exchangeable random vector $(X_1, X_2, ..., X_n)$ satisfies the MTP$_2$ property, then*

$$(t - X_{r:n} \mid X_{k:n} \leq t) \leq_{\mathrm{st}} (t - X_{r:n} \mid X_{k+1:n} \leq t),$$

for any $t \geq 0$ and $1 \leq r \leq k < n$.

For definition of MTP$_2$ functions, we refer the reader to [8]. One can show that if the MTP$_2$ assumption in Theorem 5 is removed, then the conclusion of the theorem does not remain valid [10]. Tavangar and Asadi [10] derived some other results regarding the IT of a system with exchangeable components.

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