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Some Results on Mean Vitality Function of Coherent Systems

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Abstract

In this paper, we present some results on applications of mean vitality function to comparisons of coherent systems. We also obtain an upper bound for the mean vitality function of coherent system when the lifetimes of components are independent and identically distributed.

Keywords: Coherent System, IFRA, MVF, Stochastic Orders, System Signature.

1 Introduction

Let X be a random lifetime of a system or a component having the cumulative distribution function (cdf) F with a finite moment. The mean residual life (MRL) function is defined as

$$m(t) = E(X - t|X > t) = \frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)},$$

where $\bar{F}(t) = 1 - F(t)$ is the survival (reliability) function of F. If the cdf F has the probability density function (pdf) f, then

$$m(t) = v(t) - t, \tag{1}$$

where $v(t) = E(X|X > t) = \int_t^\infty xf(x)dx/\bar{F}(t)$ is called *vitality function* (VF) or life expectancy; see, Kupka and Loo [5]. The functions VF and MRL play an important role in engineering reliability, biomedical sciences and survival analyzes; see e.g., Bairamov *et*

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al. [2], Kotz and Shanbhag [4], Ruiz and Navarro [7] and the references therein. For a continuous random variable X with the pdf f, the Shannon [10] entropy of X is defined as $H(X) = -E[\log f(X)]$ where "log" stands for the natural logarithm. Recently, Rao *et al.* [6] introduced a new measure of information, called *cumulative residual entropy* (CRE) and is defined by

$$\mathcal{E}(X) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx. \tag{2}$$

In this paper, we obtain some results about the expectation of vitality function of coherent systems. A system is said to be coherent if every component of the system is relevant and the structure function of the system is monotone. Let T denote the lifetime of a coherent system consisting of n independent and identically distributed (i.i.d.) components with lifetimes $X_1, \cdots, X_n$ which follow the common cdf F. It follows that (see e.g., Samaniego [8])

$$\bar{F}_T(t) := P(T > t) = \sum_{i=1}^{n} s_i \bar{F}_{i:n}(t), \ t > 0, \tag{3}$$

where $\bar{F}_{i:n}(t)$ is the survival function of $X_{i:n}$. The vector of coefficients $\mathbf{s} = (s_1, \cdots, s_n)$ in (3) is called the *signature* of the system where $s_i = P(T = X_{i:n})$, for $1 \leq i \leq n$, is the probability that the i-th failure causes the system failure.

2 Main results

Here, we use the concept of *mean vitality function* (MVF) order to comparisons of coherent systems based on the signature of the system. The results is considered by Toomaj and Doostparast [11].

Definition 1. *Let X and Y be random variables with finite MVF's $E(v(X))$ and $E(v(Y))$, respectively. Then X is said to be smaller than Y in the MVF order, denoted by $X \leq_{mvf} Y$, if $E(v(X)) \leq E(v(Y))$.*

Since $\mathcal{E}(X) = E(m(X))$ (see, Asadi and Zohrevand [1]), therefore, from (1) the MVF of a random variable X with finite mean $\mu = E(X)$ is

$$E(v(X)) \quad = \quad E(m(X)) + E(X) = \mathcal{E}(X) + \mu. \tag{4}$$

It can be applied the concept of MVF to comparison of coherent systems. Therefore, we have the following corollary. Let T be the lifetime of the coherent system with signature $\mathbf{s} = (s_1, \cdots, s_n)$ consisting of n i.i.d. component lifetimes $X_1, \cdots, X_n$ coming from the cdf F. Then

$$E(v(X_{1:n})) \leq E(v(T)) \leq E(v(X_{n:n})). \tag{5}$$

Corollary 2 says that the MVF of coherent systems are between the MVF's of the series and parallel systems. Hence, Expression (5) motivates the comparison of coherent systems based on MVF measure.

Example 1. Let T_1 and T_2 be lifetimes of two coherent systems with signatures $\mathbf{s}_1 = (0, \frac{3}{7}, \frac{4}{7})$ and $\mathbf{s}_2 = (0, \frac{3}{8}, \frac{5}{8})$, respectively, having $n = 3$ i.i.d. component lifetimes coming from the standard exponential distribution. It is easy to verify that $E(v(T_1)) = 2.48$ and $E(v(T_2)) = 2.55$ and hence $T_1 \leq_{mvf} T_2$. □

Now, we have the following proposition given by Toomaj and Doostparast [11]. To see the definition of usual stochastic order, we refer the reader to Shaked and Shanthikumar [9].

Let T_1 and T_2 be the lifetime of two coherent systems consisting of n i.i.d. component lifetimes from the cdfs F and G with signatures $\mathbf{s}_1$ and $\mathbf{s}_2$, respectively. If $\mathbf{s}_1 \leq_{st} \mathbf{s}_2$ and $X \leq_{st} Y$, then $T_1 \leq_{mvf} T_2$.

Example 2. *Let $\boldsymbol{s}_1 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)$ and $\boldsymbol{s}_2 = (0, 0, \frac{1}{4}, \frac{3}{4})$ be signatures of two systems consisting $n = 4$ i.i.d. components with the common cdf F. Let T_1 and T_2 be the corresponding lifetimes of the systems. It is easy to verify that $\boldsymbol{s}_1 \leq_{st} \boldsymbol{s}_2$. Then Proposition 2 implies that $T_1 \leq_{mvf} T_2$.*

In the sequel, we provide an upper bound for the MVF of a random variable by implementing some additional information. To see the definition of increasing failure rate average (IFRA), we refer the reader to Barlow and Proschan [3].

Let X be IFRA with the pdf f. Then, we have

$$\mathcal{E}(X) \leq \mu. \tag{6}$$

Proof. Since X is IFRA, it implies that

$$\left(-\frac{\log \bar{F}(t)}{t}\right)' \geq 0,\ t > 0.$$

Hence, for all $t > 0$ we have

$$-\bar{F}(t) \log \bar{F}(t) \leq t f(t),\ t > 0,$$

and the desired result follows. □

If T denote the lifetime of a coherent system consisting of n i.i.d. components which are IFRA, then it is known that T is IFRA, see Barlow and Proschan [3]. Hence from Equation (4) and Lemma 2, we have the following corollary. If T denote the lifetime of a coherent system consisting of n i.i.d. components which are IFRA, then

$$E(v(T)) \leq 2\mu_T,$$

where $\mu_T = E(T) = \sum_{i=1}^{n} s_i \mu_{i:n}$ and $\mu_{i:n}$ for $i = 1, \cdots, n$ stands for the expected lifetimes of the order statistics.

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