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Some inequalities on the relative commutativity degree

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Abstract

Let G be a finite group with subgroups H, M and $cp(H,M)$ denote the probability that an element of H commutes with an element of M. In this article, we give some inequalities and bounds for $cp(H,M)$, and study some of their consequences towards characterizing the pair (H,M) of subgroups of G.

Keywords: Pair of groups, commuting probability.

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1 Introduction

All groups in this article are finite. Let G be a group acting on a finite non-empty set X (i.e., G is represented as a group of permutations on X) and $\Pr(G,X) = |\mathrm{Fix}(G,X)|/|G||X|$, where $\mathrm{Fix}(G,X)$ is the set of all pairs (g,x) in $G \times X$ such that $gx = x$. This quantity is the probability that an element of G fixes an element of X, randomly. In the special case, if M is a normal subgroup of G and G acts on M by conjugation, then $\Pr(G,M)$ shows the probability that an element of G commutes with an element of M, in which case $cp(G,M)$ is usually used for $\Pr(G,M)$ and called the *relative commutativity degree* of M in G([1,9]). It is obvious that $cp(G,G)$ coincides with $cp(G)$ which is known as the commutativity degree or commuting probability of G (see [3,4,5,7,8]). Some known results concerning $cp(G,M)$ are as follows:

(1) If N is a normal subgroup of G with $N \subseteq M$, then $cp(G,M) \leq cp(G/N, M/N)cp(N)$. Moreover, the equality holds whenever $N \cap [G,M] = 1$ ([1; Theorem 3.9]).

(2) If the pair (G,M) is nilpotent of class 2, then $\frac{1}{[G,M]}(1 + \frac{[G,M]-1}{[M:Z(G,M)]}) \leq cp(G,M)$ ([9; Theorem 2.2(vi)]).

(3) If p be the smallest prime number dividing $|G|$, then $cp(G,M) \leq \frac{1}{p}(1 + \frac{p-1}{[M:Z(G,M)]})$ ([9; Theorem 2.2(v)]).

The concept of relative commutativity degree can be generalized as follows: Let H and M be two arbitrary subgroups of a group G. Then we define $cp(H,M)$ as the ratio

$$cp(H,M) = \frac{|\{(h,m) \in H \times M \mid [h,m] = 1\}|}{|H||M|}.$$

In this article, we present some inequalities and lower and upper bounds for $cp(H,M)$, which are vast generalization of the above results.

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2 Main Result

Given group G with a subgroup H such that $M \not\subseteq C_G(H)$, let $s_M = min\{|C_m^H| \,:\, m \in M \setminus C_G(H)\}$ and $l_M = max\{|C_m^H| \,:\, m \in M \setminus C_G(H)\}$, where $C_m^H = \{m^h \mid h \in H\}$ is the conjugacy class of m with respect to H. In the following theorem we give some bounds for $cp(H,M)$ in terms of the largest and smallest conjugacy class size of the elements of $M \setminus C_G(H)$ with respect to H.

Theorem 2.1. *If H and M are subgroups of G with $[H,M] \neq 1$, then*

$$\frac{l}{l_M}\left(1+\frac{l_M-1}{[M:M\cap C_G(H)]}\right) \leq cp(H,M) \leq \frac{1}{s_M}\left(1+\frac{s_M-1}{[M:M\cap C_G(H)]}\right).$$

In particular, if p be the smallest prime number dividing $|G|$, then

$$cp(H,M) \leq \frac{1}{p}\left(1+\frac{p-1}{[M:M\cap C_G(H)]}\right).$$

In the next result, we give another inequalities for $cp(H,M)$.

Theorem 2.2. *Let G be a group with subgroups H, M. Then*

(i) $cp(G,M) \leq cp(H,M) \leq cp(M, H\cap M) \leq cp(H\cap M)$.

(ii) If N is a normal subgroup of G with $N \subseteq H \cap M$, then

$$cp(H,M) \leq cp(H/N, M/N)cp(N).$$

(iii) $cp(H,M) \geq \frac{1}{|[H,M]|}\left(1+\frac{|[H,M]|-1}{[M:M\cap C_G(H)]}\right)$. *In particular, if $[H,M] \neq 1$ then* $cp(H,M) > \frac{1}{|[H,M]|}$.

Finally, we present some necessary and sufficient conditions under which the lower bound obtained in Theorem 2.2(*iii*) is attained.

Proposition 2.3. *Let G be a group, $H \leq G$ and $M \trianglelefteq G$ with $[H,M] \neq 1$. Then the following statements are equivalent.*

(i) $cp(H,M) = \frac{1}{|[H,M]|}\left(1+\frac{|[H,M]|-1}{[M:M\cap C_G(H)]}\right)$.

(ii) $|C_m^H| = |[H,M]|$ *for all* $m \in M - C_G(H)$.

(iii) $C_m^H = [H,M]m$ *for all* $m \in M - C_G(H)$*; in particular,* $[H,M] \subseteq C_G(H)$.

(iv) $C_H(m) \trianglelefteq H$ *and* $[H,M] \cong H/C_H(m)$ *for all* $m \in M - C_G(H)$.

(v) $[H,M] = \{hmh^{-1}m^{-1} \mid h \in H\}$ *for all* $m \in M - C_G(H)$.

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