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Dynamics of geodesic flow on $\langle z+2, -\frac{1}{z}\rangle\backslash\mathcal{H}$

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Abstract
Let $\mathcal{H} = \{z = x + iy : y > 0\}$ and $G = \langle z+2, -\frac{1}{z}\rangle$, $z \in \mathbb{C}$. This group acts on the upper half plane, $\mathcal{H}$, and the associated quotient surface is topologically a sphere with two cusps. We conjugate the geodesic flow on this surface to a special flow over the symbolic space of geometric codes associated to this flow. We will show that for $k \geq 1$, a subsystem with codes from $\mathbb{Z} \setminus \{0, \pm1, \pm2, \cdots, \pm k\}$ is a TBS. We also give bounds for the entropy of these subsystems.
Keywords: geodesic flow, geometric code, arithmetic code, topological entropy

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1 Introduction

Let $\mathcal{H} = \{z = x + iy : y > 0\}$ and $G = \langle z+2, -\frac{1}{z}\rangle$ with generators $T(z) = z + 2$ and $S(z) = -\frac{1}{z}$. The group G acts on $\mathcal{H}$ discontinuously with a Dirichlet fundamental domain $|z| \geq 1, |\mathrm{Re} z| \leq 1\}$. The associated quotient space $G\backslash\mathcal{H}$ is a finite area Riemann surface with one elliptic point, as its only singular point, and two cusps. This surface is topologically a sphere with two punctures and we denote it by M^{c2} (a sphere with two cusps). One of our goals is to study the dynamics of geodesic flow on M^{c2} which will not go to the cusps in either directions. Lifting the geodesics to TM^{c2}, the unit tangent bundle of M^{c2}, gives the geodesic flow as an invariant set on M^{c2}. In this note, we introduce geometric codes for geodesic flow on M^{c2}. Basically, these codes are bi-infinite sequences of nonzero integers which tell how a geodesic enters F infinitely many times in past and future. In fact, these codes together with the length of geodesic between two successive return of geodesic to F reveals the dynamics of geodesic flow. For then we can construct a special flow, conjugate to our flow, whose base space is the symbolic space obtained by these codes and its height function is the aforementioned length. We first introduce parameter space and then we will obtain the codes from that space. Then we give an upper and a lower bound for the topological entropy of subsystems with codes in $\mathbb{Z} \setminus \{n : |n| \leq k, k \geq 2\}$.

2 Geometric code for geodesics on M^{c2}

We apply Morse method to have the geometric codes of the geodesic flow on M^{c2}. Label the circular side of F by s and the sides $x = -1$ and $x = 1$ by t^{-1} and t respectively. We consider the oriented geodesics which enter F via side s and call them *reduced geodesics*. Any geodesic on $\mathcal{H}$ is G-equivalent to a reduced geodesic. If $\gamma = (w, u)$ is a reduced geodesic with repelling and attracting endpoints w and u respectively, then $|w| > 1$ and $|u| < 1$. By Morse method we start from an initial point of a reduced geodesic on s and move in the direction of the geodesic and count the number of times that the geodesic hits sides t or t^{-1}. A bi-infinite sequence of non-zero integers will be assigned to γ called the *geometric code* of γ. Denote the geometric code of γ by $[\gamma] = [..., n_{-1}, n_0, n_1, ...]$.

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Consider the wu coordinate in the plane. The lines $w=\pm 1$ and $u=\pm 1$ partition the plane to 9 regions. Let $\mathbb{T}_n \subset \mathbb{T}$ be the square whose opposite vertices are $(2n-1,-1)$ and $(2n+1,1)$, $n \in \mathbb{Z}\setminus\{0\}$. Now we show how the parameter space evolves geometric codes. Start with a reduced geodesic $\gamma=(w,u)\in\mathbb{T}_{n_0}$. Let $ST^{-n_0}(w,u)\in\mathbb{T}_{n_1}$ and set $\mathbb{T}_{n_0,n_1}=ST^{-n_0}(\mathbb{T}_{n_0})\cap\mathbb{T}_{n_1}$. Inductively, let $\mathbb{T}_{n_0,n_1,\dots,n_k} := ST^{-n_{k-1}}(\mathbb{T}_{n_0,n_1,\dots,n_{k-1}})\cap\mathbb{T}_{n_k}$ containing the reduced geodesic $ST^{-n_{k-1}}ST^{-n_{k-2}}...ST^{-n_0}(w,u)$. Note that $\mathbb{T}_{n_0,n_1,\dots,n_k}$ contains all geodesics having n_i as the ith entry in their geometric code, $0\le i\le k$.

Let $\mathcal{A}$ be a set of countable alphabets. Consider the space $\Sigma\subseteq\Sigma_{\mathcal{A}}=\{x=(x_i)_{i=-\infty}^{\infty},\, x_i\in\mathcal{A}\}$ and the shift map $\sigma:\Sigma\to\Sigma$ defined by $\sigma(x_i)=x_{i+1}$. The symbolic dynamical system (Σ,σ) is called two-sided *countable topological Bernoulli scheme* (TBS) if $\Sigma=\Sigma_{\mathcal{A}}$.

Let $\mathcal{B}\subseteq\mathbb{Z}\setminus\{0,\pm 1\}$ and $\Sigma_{\mathcal{B}}$ be the space of geometric codes whose alphabets are from $\mathcal{B}$.

Theorem 2.1. *The space $\Sigma_{\mathcal{B}}$ is a TBS.*

Proof. Consider the region $\mathbb{T}_{n_0}$, $n_0\in\mathcal{B}$. Since $ST^{-n_0}(\mathbb{T}_{n_0})\cap\mathbb{T}_{n_1}$ is nonempty for all $n_1\in\mathcal{B}$ it follows that $[n_0, n_1]$ is an admissible block. □

Corollary 2.2. *Let $\cdots, n_{-1}, n_0, n_1, \cdots$ be a bi-infinite sequence in $\mathbb{Z}\setminus\{0\}$ whose tails are neither all 1 nor all -1. Then this sequence represents a geometric code of an oriented geodesic on M^{c2} not going to cusps in positive or negative times.*

3 Entropy

In this section we use the method in [1] to give bounds for topological entropy on our subsystems which are all TBS. The subsystems we have chosen are those whose alphabets are in $\mathbb{Z}\backslash\{0,\pm1,\pm2,...,\pm k\}$ and those with alphabets in $\mathbb{N}\backslash\{1,2,...,k\}$, $k\ge 2$.

Define the family $T_{\ell,\Sigma}=\{T^s_{\ell,\Sigma}\}_{s\in\mathbb{R}}$ to be the *special flow* constructed over the *base space* Σ and *height function* ℓ. For $k\in\mathbb{N}\setminus\{1\}$, let $\mathcal{A}_k=\{n: |n|\ge k\}$ and $h(T_{\ell,\Sigma_{\mathcal{A}_k}})$ the topological entropy of $T_{\ell,\Sigma_{\mathcal{A}_k}}$.

Theorem 3.1. *Let $\zeta(\cdot)$ be the Riemann zeta function. Then $x_l < h(T_{\ell,\Sigma_{\mathcal{A}_k}}) < x_u$ where for $\alpha\in\{l,u\}$, x_α is the unique solution of*

$$2c_\alpha^{-2x}\left(\zeta(2x)-\sum_{n=1}^{k-1}\frac{1}{n^{2x}}+\frac{1}{k^{2x}}\right)=1. \tag{1}$$

Here $c_u=2-\frac{1}{k[2k]}=2-\frac{1}{k(k+\sqrt{k^2-1})}$ and $c_l=2+\frac{1}{k[2k]}$.

Theorem 3.2. *Let $x=[\gamma]$ be the geometric code of γ with repelling and attracting points $w=w(x)$ and $u=u(x)$ respectively. Then $\ell(x)=2\ln(w(x))+\ln(g(x))-\ln(g(\sigma x))$ where $g(x)=\frac{(w(x)-u(x))\sqrt{w(x)^2-1}}{w(x)^2\sqrt{1-u(x)^2}}$.*

Proof. With almost no change, the lines of proof is similar to the proof of [2, Theorem 4Dawoud Ahmadi Dastjerdi and Sanaz Lamei]. Just let z_1 and z_1' be the intersection of $\gamma=(w,u)$ with $|z|=1$ and $|z-2n_1|=1$ respectively. □

For any subsystem $\Sigma_{\mathcal{B}}$ of Σ denote the positive continued functions like $f(x)$ depending on the zero coordinate and satisfying the condition $\sum_{k=1}^{\infty}f(\sigma^k(x))=\sum_{k=1}^{\infty}f(\sigma^{-k}(x))=\infty$ by $\mathcal{F}_0(\Sigma_{\mathcal{B}})$.

Let H be a directed graph with vertex set $V=\mathcal{A}$ and the edge set E. The path $\tau=(v_0,...,v_n)$ is called a *simple v-cycle* if $v_0=v_n=v$ and $v_i\ne v$ for $1\le i\le n-1$. Let $C(H;v)$ be the set of all simple v-cycles in the graph H. Let $f\in\mathcal{F}_0(\Sigma_{\mathcal{B}})$ and $F_{f,V}(x)=\sum_{v\in V}x^{f(v)}$ be a series for $x\ge 0$ and set $\phi_{H,f,w}(x)=\sum_{\tau\in C(H;w)}x^{f^*(\tau)}$, $x\ge 0$ be the *generating function* with respect to the special flow $T_{f,\Sigma}$ where $f^*(\tau)=\sum_{i=0}^{n}f(v_i)$, $\tau=(v_0,...,v_n)$.

Remark 3.3. *Let $(\Sigma_{\mathcal{B}}, \sigma)$ be a 1-step topological Markov chain and $f \in \mathcal{F}_0(\Sigma_{\mathcal{B}})$. Then by [1, Remark 1Dawoud Ahmadi Dastjerdi and Sanaz Lamei], $h(T_{f,\Sigma}) = -\ln(\hat{x}_f)$ where $\hat{x}_f$ is either the unique solution of $\phi_{H,f,v}(x) = 1$ or $\hat{x}_f = r(\phi_{H,f,v})$.*

Lemma 3.4. *Let $c > 1$ and $f(x) = 2\ln(cn_0)$, $|n_0| \geq k \geq 2$. Then $h(T_{f,\Sigma_{\mathcal{A}_k}}) = -\hat{x}_f$ where $\hat{x}_f$ is the unique solution of $\phi_{H_{\mathcal{A}_k},f,v_k}(x) = 1$.*

Proof. Since $f(x) = 2\ln(cn_0)$ and $|n_0| \geq k$ so $f \in \mathcal{F}_0(\Sigma_{\mathcal{A}_k})$. Let $H_k := H_{\mathcal{A}_k}$ be a complete graph with vertex set $V(H_k) = \mathcal{A}_k$ and edge set $E(H_k)$. Apply [1, Lemma 1Dawoud Ahmadi Dastjerdi and Sanaz Lamei] for $m = 1$ to have $\phi_{H_k,f,v_k}(x) = \alpha_{00}(x) + \alpha_{10}(x)A_1(x)$. This implies $h(T_{f,\Sigma_{\mathcal{A}_k}}) = -\ln(\hat{x}_f)$ where $\hat{x}_f$ is either the unique solution of $\phi_{H_k,f,v_k}(x) = 1$ or $\hat{x}_f = r(\phi_{H_k,f,v_k}) = r(A_1)$. We want to show that for our case $\hat{x}_f$ is the unique solution of $\phi_{H_k,f,v_k}(x) = 1$. For $0 \leq x < r(F_{f,V(H_k)})$ set

$$\tilde{x}_0 = \begin{cases} r(F_{f,V(H_k)}), & \text{if } M(x) \text{ is invertible} \\ \inf\{x : 0 \leq x < r(F_{f,V(H_k)}), \det M(x) = 0\}, & \text{otherwise.} \end{cases} \tag{2}$$

From [1, Theorem 2Dawoud Ahmadi Dastjerdi and Sanaz Lamei] we have $\lim_{x \to \tilde{x}_0^-} \phi_{H_k,f,v_k}(x) = \infty$ which means $\phi_{H_k,f,v_k}(x) = 1$ has a solution in $0 < x < r(F_{f,V(H_k)})$. We will show that this is indeed the case. We achieve this if $\det M(x) = 0$ in $0 < x < r(F_{f,V(H_k)})$. But for $T_{f,\Sigma_{\mathcal{A}_k}}$, $\det M(x) = 1 - 2\sum_{n=k}^{\infty} x^{2\ln cn}$. So by setting $\ln\frac{1}{x} = s$, we have

$$\frac{c^{2s}}{2} = \sum_{n=k}^{\infty} \frac{1}{n^{2s}}. \tag{3}$$

Now (3) has a unique solution on $\frac{1}{2} < s < \infty$ or $M(x)$ has a unique solution on $0 < x < \frac{1}{\sqrt{e}}$. $\square$

Proof of Theorem 1a. For $x = (..., n_0, n_1, ...)$

$$c_u|n_0| = |2n_0| - \frac{1}{[2k]} \leq |w(x)| = |2n_0 - \cfrac{1}{2n_1 - \cfrac{1}{2n_2 - \cfrac{1}{\ddots}}}| \leq |2n_0| + \frac{1}{[2k]} = c_l|n_0|, \tag{4}$$

where $c_l = 2 + \frac{1}{k[2k]}$ and $c_u = 2 - \frac{1}{k[2k]}$. Let $f_\alpha(x) = 2\ln c_\alpha|n_0|$ where $\alpha \in \{l, u\}$. Then by Abramov formula, $h(T_{f_l,\Sigma_{\mathcal{A}_k}}) \leq h(T_{f,\Sigma_{\mathcal{A}_k}}) \leq h(T_{f_u,\Sigma_{\mathcal{A}_k}})$.

We have $\phi_{H_k,f,v_k}(x) = \frac{x^{f(v)}}{1 - x^{f(v)} - F_{f,V(H_k)}(x)}$ when $1 - x^{f(v)} - F_{f,V(H_k)}(x) > 0$ [4] and H_k is the complete graph introduced in the proof of Lemma 13. See also [1, Remark 2Dawoud Ahmadi Dastjerdi and Sanaz Lamei]. By the above lemma, $\hat{x}_l$ is the unique solution of $\phi_{H_k,f_l,v_k}(x) = 1$ and by letting $x_l = -\ln\hat{x}_l$, we have x_l is the solution of

$$2c_l^{-2x}\left(\zeta(2x) - \sum_{n=1}^{k-1}\frac{1}{n^{2x}} + \frac{1}{k^{2x}}\right) = 1.$$

$\square$

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