



Higher-order asymptotic formula for the eigenvalues of Sturm-Liouville problem with indefinite weight function in the case of $y(a) = y'(b) = 0, y'(a) = y(b) = 0$

F. Dastmalchi Saei

Department of Mathematics, Tabriz
Branch, Islamic Azad University,
Tabriz, Iran

Abstract

In this paper, we investigate the asymptotic behavior of the differential equation

$$y'' + (\lambda r(x) - q(x))y = 0, \quad 0 \leq x \leq 1,$$

where $[0, 1]$ contains a finite number of zeros of $r(x)$, λ is a real parameter and the function $q(x)$ is bounded and integrable in $[0, 1]$. Using a technique used previously in [7], we derive the higher-order asymptotic distribution of the positive eigenvalues associated with this equation for the SLP (in the boundary condition of, $y(0) = y'(1) = 0$ or $y'(0) = y(1) = 0$).

Keywords: Sturm-Liouville problem, eigenvalues, asymptotic

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1 Introduction

We study the indefinite Sturm-Liouville spectral problem

$$y'' + (\lambda r(x) - q(x))y = 0, \quad a \leq x \leq b \quad (1)$$

$$y'(a) = y(b) = 0 \vee y(a) = y'(b) = 0,$$

defined on the interval $[a, b]$ where λ is a real parameter, $r(x), q(x)$ are real and integrable on $[a, b]$; moreover,

$$\int_a^b \sqrt{r_+(t)} dt > 0, \quad \text{where} \quad r_+(x) = \max\{r(x), 0\}. \quad (2)$$

It follows from [2] that the spectrum of this problem is discrete and has no finite accumulation points; moreover, only finitely many eigenvalues lie the outside the real and imaginary axes. In what follows, we shall assume that λ is a positive parameter. In [2] it was shown that the asymptotics of the eigenvalues is of the form

$$\lambda_n \sim \frac{n\pi}{\int_a^b \sqrt{r_+(t)} dt}.$$

Our goal is to refine the asymptotics under the additional assumptions of smoothness of the functions $r(x)$ and $q(x)$. In addition, we assume that $r(x)$ has a finite number of zeros, which are called



turning points.

The outline of our paper is as follows. First, we find the asymptotics of eigenvalues for one turning point. Next, using a technique previously in [7], we derive the higher order asymptotic distribution of the positive eigenvalues in the case of two turning points. Finally, in the case of an arbitrary finite number of turning points it can be reduced to the two cases discussed above.

2.The case of one turning point

First, consider the case

$$r(x) = (x - x_\nu)^{l_\nu} h(x), \quad h(x) > 0.$$

To simplify the formulas, we assume that x varies on the closed interval with endpoints a and b , where $r(a) < 0$ and $r(b) > 0$. The turning point x_ν lies between a and b .

We distinguish four different types of turning points:

$$T_\nu = \begin{cases} I & \text{if } l_\nu \text{ is even and } r(x) < 0 \text{ in } [a, b] \\ II & \text{if } l_\nu \text{ is even and } r(x) > 0 \text{ in } [a, b] \\ III & \text{if } l_\nu \text{ is odd and } r(x) < 0 \text{ in } [x_\nu, b] \\ IV & \text{if } l_\nu \text{ is odd and } r(x) > 0 \text{ in } [x_\nu, b] \end{cases}$$

is called of type x_ν . By Langer's transformation we can make zero of $r(x)$ the origin. To be specific, let us define the Langer's transformation $\xi(x)$ for different type of TP.

For a turning point of Type I :

$$\xi_I(x) = \begin{cases} -\left\{ \int_x^{x_\nu} (-r)^{1/2}(t) dt \right\}^{\frac{2}{\ell+2}} & x \leq x_\nu \\ -\left\{ \int_{x_\nu}^x (-r(t))^{1/2} dt \right\}^{\frac{2}{\ell+2}} & x_\nu \leq x. \end{cases}$$

For a turning point of Type II :

$$\xi_{II}(x) = \begin{cases} \left\{ \int_x^{x_\nu} r^{1/2}(t) dt \right\}^{\frac{2}{\ell+2}} & x \leq x_\nu \\ \left\{ \int_{x_\nu}^x r(t)^{1/2} dt \right\}^{\frac{2}{\ell+2}} & x_\nu \leq x. \end{cases}$$

For a turning point of Type III :

$$\xi_{III}(x) = \begin{cases} \left\{ \int_x^{x_\nu} r^{1/2}(t) dt \right\}^{\frac{2}{\ell+2}} & x \leq x_\nu \\ -\left\{ \int_{x_\nu}^x (-r(t))^{1/2} dt \right\}^{\frac{2}{\ell+2}} & x_\nu \leq x. \end{cases}$$

For a turning point of Type IV :

$$\xi_{IV}(x) = \begin{cases} -\left\{ \int_x^{x_\nu} (-r(t))^{1/2} dt \right\}^{\frac{2}{\ell+2}} & x \leq x_\nu \\ \left\{ \int_{x_\nu}^x (r(t))^{1/2} dt \right\}^{\frac{2}{\ell+2}} & x_\nu \leq x. \end{cases}$$

From [9] we rewrite showing the connection between the argument of complex valued solution of (14) in the interval containing one of the turning point say, x_ν , and the argument of complex valued solution of Sturm-Liouville equation with one turning point in $x = 0$ in the same interval. In fact the following result illustrates a crucial relationship between a general problem (14) with a turning point at x_ν to a transformed problem in which is mapped to $x = 0$. We show that such a transformation preserves the argument of any fixed complex valued solution.

Theorem 1.1. *Let z be a strictly complex -valued solution of the differential equations*

$$y'' + (\rho^2 r(x) - q(x))y = 0, \quad x \in [0, 1] \tag{3}$$

and W be a solution of

$$W'' + (u^2(-1)^{M_\nu} \xi^{\ell_\nu} - R_\nu(\xi))W = 0, \quad \xi \in [c, d] \tag{4}$$



then on the interval $[x_\nu - \epsilon, x_\nu + \epsilon]$

$$\arg W(\xi(x)) = \arg z,$$

where $r(x) = \prod_{j=1}^n (x - x_j)^{l_j} \phi_0(x)$ and

$$R_\nu(\xi) = \left(\frac{dx}{d\xi}\right)^{1/2} \frac{d^2}{d\xi^2} \left\{ \frac{1}{\left(\frac{dx}{d\xi}\right)^{1/2}} \right\} + \left(\frac{dx}{d\xi}\right)^2 q(x(\xi)).$$

$M_k =$ the number of turning of type (III) or (IV) in $(x_k, 1)$, or one can see that $(-1)^{M_k} = (-1)^{l_n + \dots + l_{k-1}}$, $c < 0 < d$, $u^2 = \frac{(\ell_\nu + 2)^2}{4} \rho^2$, the transformation $\xi(x)$ is Langer's transformation.

Proof: For proof see [9].

3. The main result

We begin by consolidating some results from [5, 9] for completeness. For a complex-valued solution $\Omega(x, \lambda)$, of

$$y'' + \lambda x^\alpha y = 0, \quad (E_0)$$

we form the logarithmic derivative $r_0(x, \lambda) = \Omega'(x, \lambda)/\Omega(x, \lambda)$, a quantity that exists for each $x \in [a, b]$ since the real and imaginary parts of Ω are linearly independent solution of (E_0) . The quantity $r_1(x, \lambda)$ is defined by setting

$$r_1(x, \lambda) = - \int_x^b q(t) e^{2 \int_x^t r_0(s, \lambda) ds} dt,$$

while the $r_n(x, \lambda)$ are defined recursively (for $n \geq 1$) by

$$r_{n+1}(x, \lambda) = \int_x^b r_n^2(x, \lambda) \exp\left(2 \sum_{l=0}^n \int_x^t r_l(s, \lambda) ds\right) dt$$

It follows (cf. [4]) that the function

$$r(x, \lambda) = \sum_{n=0}^{\infty} r_n(x, \lambda) := S(x, \lambda) + iT(x, \lambda)$$

is a series solution (in x) of the Riccati equation

$$v' = q - \lambda x^\alpha - v^2$$

from which one can reconstruct solutions of (14) with Neumann condition $y'(a) = y'(b) = 0$ via the following result:

Theorem 1.2. (see Harris-Talarico [4]) There exists λ_0 such that any real valued solution of

$$y'' + (\lambda x^\alpha - q(x))y = 0 \tag{5}$$

can be expressed as :

$$Z(x, \lambda) = c_1 e^{\int_a^x S(t, \lambda) dt} \cos\left(c_2 + \int_a^x T(t, \lambda) dt\right)$$

for $x \in [a, b]$ ($a < 0 < b$) and $|\lambda| \geq \lambda_0$ where $c_1, c_2 \in \mathfrak{R}$. If $Z(\cdot, \lambda)$ satisfies

$$y(a) \cos \gamma + y'(a) \sin \gamma = 0 \tag{6}$$

then

$$\begin{aligned} c_2 =: c_2^a &= \frac{\pi}{2} && \text{if } \gamma = 0 \\ &= \arctan\left(\frac{1}{T(a, \lambda)} S(a, \lambda) + \cot \gamma\right) && \text{if } \gamma \neq 0 \end{aligned} \tag{7}$$



Similarly, if Z satisfies

$$y(b) \cos \beta + y'(b) \sin \beta = 0 \tag{8}$$

then

$$\begin{aligned} c_2 =: c_2^b &= n\pi + \frac{\pi}{2} && \text{if } \beta = 0 \\ &= n\pi + \arctan\left(\frac{1}{T(b, \lambda)} S(b, \lambda) + \cot \beta\right) && \text{if } \beta \neq 0 \end{aligned} \tag{9}$$

for all integer n .

It follows from (7) and (9) that the eigenvalues of (7), (6) and (8), i.e., our problem (14), are the values of λ for which

$$c_2^a + \int_a^b T(t, \lambda) dt = c_2^b \tag{10}$$

We see from [3], that the asymptotic distribution of the eigenvalues of (7), (6) and (8) is therefore determined by the following transcendental equation:

$$\begin{aligned} n\pi + \arctan\left(\frac{1}{T(b, \lambda)} S(b, \lambda) + \cot \beta\right) &= \int_a^b T(t, \lambda) dt + \arctan\left(\frac{1}{T(a, \lambda)} S(a, \lambda) + \cot \gamma\right) \\ &= \Im \int_a^b r(t, \lambda) dt + \arctan\left(\frac{1}{T(b, \lambda)} S(b, \lambda) + \cot \gamma\right) \\ &= \Im \left(\int_a^b r_0(t, \lambda) dt + \int_a^b r_1(t, \lambda) dt + \dots \right) \\ &+ \arctan\left(\frac{1}{T(a, \lambda)} S(a, \lambda) + \cot \gamma\right) \\ &= \arg \Omega(b, \lambda) - \arg \Omega(a, \lambda) - \frac{\pi}{2k} \int_0^b xq(x) J_\nu^2(k^{-1} \lambda^{1/2} x^k) dx + \dots \\ &+ \arctan\left(\frac{1}{T(a, \lambda)} S(a, \lambda) + \cot \gamma\right). \end{aligned} \tag{11}$$

Note that we use the following result from [8],

$$\Im \int_a^b r(t, \lambda) dt = \arg \Omega(b, \lambda) - \arg \Omega(a, \lambda) - \frac{\pi}{2k} \int_0^b xq(x) J_\nu^2(k^{-1} \lambda^{1/2} x^k) dx.$$

By applying the above relation to approximate eigenvalues in the case of $\gamma = 0, \beta = \frac{\pi}{2}$ or $\gamma = \frac{\pi}{2}, \beta = 0$.

Theorem 1.3. Consider the differential equation (1) on $[a, b]$ under condition (2). Then the positive eigenvalues admit the following asymptotic representation:

Case1: $\gamma = 0, \beta = \frac{\pi}{2}$

(a) Let x_ν be of type IV. Then

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)} dt} - \frac{1}{n\pi} \left(\frac{4(\nu - 1)^2 - 1}{8 \int_a^b \sqrt{r_+(t)} dt} - \frac{1}{2} H(b) \right) + O\left(\frac{1}{n^2}\right)$$

where

$$H(b) = \int_{x_\nu}^b \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}} \frac{d^2}{dx^2} (\tilde{r}^{-1/4}) \right) \frac{\tilde{r}}{r^{1/2}} dx,$$



and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

(b) Let x_ν be of type III. Then

$$\sqrt{\lambda_n} = \frac{n\pi + \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)} dt} - \frac{1}{n\pi} \left(\frac{4(\nu-1)^2 - 1}{8 \int_a^b \sqrt{r_+(t)} dt} - \frac{1}{2} H(a) \right) + O\left(\frac{1}{n^2}\right)$$

where

$$H(a) = \int_a^{x_\nu} \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}} \frac{d^2}{dx^2} (\tilde{r}^{-1/4}) \right) \frac{\tilde{r}}{r^{1/2}} dx,$$

and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

(c) Let x_ν be of type II. Then

$$\sqrt{\lambda_n} = \frac{n\pi}{\int_a^{x_\nu} \sqrt{r(t)} dt} - \frac{1}{n\pi} \left\{ \frac{4(\nu-1)^2 - 1}{8 \int_a^{x_\nu} \sqrt{r(t)} dt} + \left\{ \frac{4(\nu-1)^2 - 1}{8 \int_{x_\nu}^b \sqrt{r(t)} dt} - \frac{1}{2} H(a) - \frac{1}{2} H(b) \right\} \right\} + O\left(\frac{1}{n^2}\right)$$

where $H(a)$ and $H(b)$ are defined above. Case1: $\gamma = 0, \beta = \frac{\pi}{2}$

(a) Let x_ν be of type IV. Then

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)} dt} - \frac{1}{n\pi} \left(\frac{4(\nu-1)^2 - 1}{8 \int_a^b \sqrt{r_+(t)} dt} - \frac{1}{2} H(b) \right) + O\left(\frac{1}{n^2}\right)$$

where

$$H(b) = \int_{x_\nu}^b \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}} \frac{d^2}{dx^2} (\tilde{r}^{-1/4}) \right) \frac{\tilde{r}}{r^{1/2}} dx,$$

and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$

(b) Let x_ν be of type III. Then

$$\sqrt{\lambda_n} = \frac{n\pi + \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)} dt} - \frac{1}{n\pi} \left(\frac{4(\nu-1)^2 - 1}{8 \int_a^b \sqrt{r_+(t)} dt} - \frac{1}{2} H(a) \right) + O\left(\frac{1}{n^2}\right)$$

where

$$H(a) = \int_a^{x_\nu} \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}} \frac{d^2}{dx^2} (\tilde{r}^{-1/4}) \right) \frac{\tilde{r}}{r^{1/2}} dx,$$

and

$$\tilde{r} = \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}.$$



(c) Let x_ν be of type II. Then

$$\sqrt{\lambda_n} = \frac{n\pi}{\int_a^b \sqrt{r(t)} dt} - \frac{1}{n\pi} \left\{ \frac{4(\nu-1)^2-1}{8 \int_a^{x_\nu} \sqrt{r(t)} dt} + \left\{ \frac{4(\nu-1)^2-1}{8 \int_{x_\nu}^b \sqrt{r(t)} dt} - \frac{1}{2}H(a) - \frac{1}{2}H(b) \right\} + O\left(\frac{1}{n^2}\right) \right\}$$

where $H(a)$ and $H(b)$ are defined above.

Proof: Let us proof only case (1a), from [14] for $\alpha = 4n \pm 1$ and $x < 0$ we know that $\Omega(x, \lambda) = e^{i\pi/2}|x|^{1/2}H_\nu^{(1)}(e^{ik\pi}\rho|x|^k)$, and from [1, (9.2.7)], Hankel's expansion the form

$$H_\nu^{(1)}(z) = \sqrt{\frac{2}{\pi z}} e^{i(z - \frac{1}{2}\nu\pi - \frac{\pi}{4})} M(z), \quad (-\pi < \arg z < \pi),$$

where $M(z)$ is bounded away from the origin and $M(z) \rightarrow 1$ as $|z| \rightarrow \infty$ in the sector.

Now since $\Omega(x, \lambda) = i\rho^{-\nu}e^{-ik\pi\nu}(e^{ik\pi}\rho|x|^k)^\nu H_\nu^{(1)}(e^{ik\pi}\rho|x|^k)$, where $k = \frac{\alpha+2}{2}, \nu = \frac{1}{\alpha+2}$ and $k\nu = \frac{1}{2}$, by using ([1],(9.1.30)) we find $\Omega'(x, \lambda) = ik\rho e^{ik\pi}\rho|x|^{k-3/2}xH_{\nu-1}^{(1)}(e^{ik\pi}\rho|x|^k)$. Therefore

$$\begin{aligned} \frac{\Omega'(x, \lambda)}{\Omega(x, \lambda)} &= k\rho e^{ik\pi}x|x|^{k-2} \frac{H_{\nu-1}^{(1)}(z)}{H_\nu^{(1)}(z)} \\ &= k\rho e^{ik\pi}x|x|^{k-2} \left(i + \frac{8\nu-4}{8z} + O\left(\frac{1}{z^2}\right) \right), \quad z = e^{ik\pi}\rho|x|^k. \end{aligned}$$

Then

$$\Re \frac{\Omega'(x, \lambda)}{\Omega(x, \lambda)} = k\rho x|x|^{k-2} - k \frac{8\nu-4}{8|x|} + O\left(\frac{1}{z^2}\right).$$

and

$$\Im \frac{\Omega'(x, \lambda)}{\Omega(x, \lambda)} = 0.$$

On the other hand,

$$\frac{S(a, \lambda)}{T(a, \lambda)} = \frac{\Re r_0(a, \lambda) + \Re r_1(a, \lambda) + \dots}{\Im r_0(a, \lambda) + \Im r_1(a, \lambda) + \dots} = \frac{\Re \frac{\Omega'(a, \lambda)}{\Omega(a, \lambda)} + \Re \left\{ - \int_a^b q(t) e^{2 \int_a^t r_0(s, \lambda) ds} dt \right\} + \dots}{\Im \frac{\Omega'(a, \lambda)}{\Omega(a, \lambda)} + \Im \left\{ - \int_a^b q(t) e^{2 \int_a^t r_0(s, \lambda) ds} dt \right\} + \dots}.$$

Now we have:

$$\begin{aligned} \int_a^b q(t) e^{2 \int_a^t r_0(s, \lambda) ds} dt &= \int_a^0 q(t) e^{2 \int_a^t r_0(s, \lambda) ds} dt + \int_0^b q(t) e^{2 \int_a^0 r_0(s, \lambda) ds + 2 \int_0^t r_0(s, \lambda) ds} dt \\ &= \frac{1}{|\Omega(a, \lambda)|^2} \int_a^0 q(t) |\Omega(t, \lambda)|^2 e^{2i\{\arg \Omega(t, \lambda) - \arg \Omega(a, \lambda)\}} dt \\ &\quad + \int_0^b q(t) \left| \frac{\Omega(0, \lambda)}{\Omega(a, \lambda)} \right|^2 e^{2i\{\arg \Omega(0, \lambda) - \arg \Omega(a, \lambda)\}} \times \left| \frac{\Omega(t, \lambda)}{\Omega(0, \lambda)} \right|^2 e^{2i\{\arg \Omega(t, \lambda) - \arg \Omega(0, \lambda)\}} dt \\ &= \frac{e^{-2i \arg \Omega(a, \lambda)}}{|\Omega(a, \lambda)|^2} \left\{ \int_a^0 q(t) |\Omega(t, \lambda)|^2 e^{2i \arg \Omega(t, \lambda)} dt + \int_0^b q(t) |\Omega(t, \lambda)|^2 e^{2i \arg \Omega(t, \lambda)} dt \right\} = O(1). \end{aligned}$$

Similarly for $\alpha = 4n \mp 1$ and $x > 0$ we know that $\Omega(x, \lambda) = x^{1/2}H_\nu^{(1)}(\rho x^k)$. Now since $\Omega(x, \lambda) = \rho^{-\nu}(\rho x^k)^\nu H_\nu^{(1)}(\rho x^k)$, by using ([1],(9.1.30)) we find $\Omega'(x, \lambda) = kx^{k-1/2}\rho H_{\nu-1}^{(1)}(\rho x^k)$. Therefore

$$\frac{\Omega'(x, \lambda)}{\Omega(x, \lambda)} = k\rho x^{k-1} \frac{H_{\nu-1}^{(1)}(z)}{H_\nu^{(1)}(z)} = k\rho x^{k-1} \left(i + \frac{8\nu-4}{8z} + O\left(\frac{1}{z^2}\right) \right).$$



Then

$$\Re \frac{\Omega'(x, \lambda)}{\Omega(x, \lambda)} = k\rho x^{k-1} \frac{8\nu - 4}{8\rho x^k} = k \frac{8\nu - 4}{8x} + O\left(\frac{1}{z^2}\right),$$

and

$$\Im \frac{\Omega'(x, \lambda)}{\Omega(x, \lambda)} = k\rho x^{k-1} + O\left(\frac{1}{z^2}\right),$$

whence

$$\Re \frac{\Omega'(b, \lambda)}{\Omega(b, \lambda)} = k \frac{8\nu - 4}{8b} + O\left(\frac{1}{z^2}\right),$$

and

$$\Im \frac{\Omega'(b, \lambda)}{\Omega(b, \lambda)} = k\rho b^{k-1} = \sqrt{\lambda} b^{k-1} + O\left(\frac{1}{z^2}\right),$$

also

$$\Re r_1(b, \lambda) = \Im r_1(b, \lambda) = 0.$$

By substitution in (11) and using :

$$\arctan x = x - \frac{x^3}{3} + \dots,$$

we have (the particulars of the calculations are omitted):

$$\sqrt{\lambda_n} = \frac{n\pi - \frac{\pi}{4}}{\int_a^b \sqrt{r_+(t)} dt} - \frac{1}{n\pi} \left(\frac{4(\nu - 1)^2 - 1}{8 \int_a^b \sqrt{r(t)} dt} - \frac{1}{2} H(b) \right) + O\left(\frac{1}{n^2}\right).$$

2 Main Result

The cases of two and n turning points

From now then, without losing generalization, we suppose that the coefficients $q(x)$ and $r(x)$ satisfy:

- (i) $r(x)$ is real and has in $[0, 1]$ n zeros x_ν of order $l_\nu \in \mathbb{N}, 1 \leq \nu \leq n$ where $0 < x_1 < x_2 < \dots < x_n < 1$.
- (ii) The function $\phi_0 : I \rightarrow \mathbb{R} - \{0\}, x \rightarrow r(x) \prod_{\nu=1}^n (x - x_\nu)^{-l_\nu}$ is twice continuously differentiable.
- (iii) $q(x)$ is bounded and integrable in I .

We shall use the symbol $\Omega_{IV}(\xi, u)$ to signify the complex-valued solution of

$$W'' + u^2 \xi^{l_\nu} W = 0,$$

where ξ is corresponding Langer's transformation of turning point of type IV. We will use the symbols $\Omega_I(\xi, u), \Omega_{II}(\xi, u)$ and $\Omega_{III}(\xi, u)$ in similar case.

Now we can derive the following results on the distribution of the eigenvalues of (14) with Neumann boundary condition .

We consider only the following case:

$$r(0) < 0, r(1) < 0.$$

1.a $T_1 = IV, T_2 = III$.

We suppose that the weight function $r(x)$ has in $[0, 1]$ two zeros x_1 and x_2 where x_1 of type IV and x_3 of type III. By (11) the distribution of positive eigenvalue satisfies :



$$\begin{aligned}
 n\pi &= \Im \int_0^1 \frac{y'}{y} dx = \Im \left(\int_0^{\alpha_{12}} \frac{y'}{y} dx + \int_{\alpha_{12}}^1 \frac{y'}{y} dx \right) + \arctan\left(\frac{S(0, \lambda)}{T(0, \lambda)}\right) - \arctan\left(\frac{S(1, \lambda)}{T(1, \lambda)}\right) \\
 &= \rho \int_0^1 \sqrt{r_+(t)} dt + \frac{\pi}{2} - \frac{1}{4\rho} \left(\frac{8\nu_1 - 4}{\int_{x_1}^{x_2} \sqrt{r(t)} dt} + \frac{8\nu_2 - 4}{\int_{x_1}^{x_2} \sqrt{r(t)} dt} \right) + \frac{1}{4\rho} \left(\frac{4\nu_1^2 - 1}{\int_{x_1}^{x_2} \sqrt{r(t)} dt} + \frac{4\nu_2^2 - 1}{\int_{x_1}^{x_2} \sqrt{r(t)} dt} \right) \\
 &\quad - \frac{1}{2u} P(x_1, x_2) + O\left(\frac{1}{u^2}\right),
 \end{aligned} \tag{12}$$

where $\alpha_{12} \in (x_1, x_2)$ is such that $\int_{x_1}^{\alpha_{12}} \sqrt{r(t)} dt = \int_{\alpha_{12}}^{x_2} \sqrt{r(t)} dt$ (the existence of α_{12} follows by Intermediate Value Theorem), and

$$P(x_1, x_2) = P_{IV}(x_1, \alpha_{12}) + P_{III}(x_2, \alpha_{12}) = \int_{x_1}^{\alpha_{12}} E_{IV}(x) dx + \int_{\alpha_{12}}^{x_2} E_{III}(x) dx,$$

$$P_{IV}(x_1, \alpha_{12}) = \int_0^{d_{12}} \frac{R_{IV}(\xi)}{\xi^{\frac{l}{2}}} d\xi = \int_{x_1}^{\alpha_{12}} \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}} \frac{d^2}{dx^2} (\tilde{r}^{-1/4}) \right) \frac{\tilde{r}}{r^{\frac{1}{2}}} dx := \int_{x_1}^{\alpha_{12}} E_{IV}(x) dx,$$

and

$$\begin{aligned}
 \tilde{r} &= \left(\frac{d\xi}{dx}\right)^2 = \frac{4r(x)}{(l+2)^2(\xi(x))^l}, \xi(\alpha_{12}) = d_{12}, \xi(x_1) = 0, \xi(0) = c_{12} \\
 P_{III}(x_2, \alpha_{12}) &= \int_{\alpha_{12}}^{x_2} \left(\frac{q(x)}{\tilde{r}(x)} - \frac{1}{\tilde{r}^{3/4}} \frac{d^2}{dx^2} (\tilde{r}^{-1/4}) \right) \frac{\tilde{r}}{r^{\frac{1}{2}}} dx := \int_{\alpha_{12}}^{x_2} E_{III}(x) dx.
 \end{aligned}$$

By inversion, we get

$$\rho_n = \frac{n\pi - \frac{\pi}{2}}{\int_0^1 \sqrt{r_+(t)} dt} - \frac{1}{n\pi} \left\{ \frac{[4(\nu_1 - 1)^2 - 1] + [4(\nu_2 - 1)^2 - 1]}{4 \int_{x_1}^{x_2} \sqrt{r(t)} dt} - \frac{1}{2} P(x_1, x_2) \right\} + O\left(\frac{1}{n^2}\right).$$

2.a $T_1 = IV, T_2 = T_3 = \dots = T_{n-1} = II, T_n = III$.

By applying the same method and using theorem (1),(2) we get:

$$\begin{aligned}
 n\pi &= \rho \int_0^1 \sqrt{r_+(t)} dt - \frac{(n-1)\pi}{2} + \frac{n\pi}{2} + \frac{1}{4\rho} \left(\frac{4(\nu_1 - 1)^2 - 1}{\int_{x_1}^{x_2} \sqrt{r(t)} dt} + \frac{4(\nu_2 - 1)^2 - 1}{\int_{x_1}^{x_2} \sqrt{r(t)} dt} + \frac{4(\nu_2 - 1)^2 - 1}{\int_{x_2}^{x_3} \sqrt{r(t)} dt} \right. \\
 &\quad + \frac{4(\nu_3 - 1)^2 - 1}{\int_{x_2}^{x_3} \sqrt{r(t)} dt} + \frac{4(\nu_3 - 1)^2 - 1}{\int_{x_3}^{x_4} \sqrt{r(t)} dt} + \dots + \frac{4(\nu_{n-1} - 1)^2 - 1}{\int_{x_{n-2}}^{x_{n-1}} \sqrt{r(t)} dt} + \frac{4(\nu_{n-1} - 1)^2 - 1}{\int_{x_{n-1}}^{x_n} \sqrt{r(t)} dt} \\
 &\quad \left. + \frac{4\nu_n^2 - 1}{\int_{x_{n-1}}^{x_n} \sqrt{r(t)} dt} \right) - \frac{1}{2u} P(x_1, x_2, \dots, x_n) + O\left(\frac{1}{u^2}\right),
 \end{aligned}$$

where

$$P(x_1, x_2, \dots, x_n) = \int_{x_1}^{\alpha_{12}} E_{IV}(x) dx + \sum_{i=1}^{n-2} \int_{\alpha_{i(i+1)}}^{x_{i+1}} E_{II}^-(x) dx + \sum_{i=1}^{n-2} \int_{x_{i+1}}^{\alpha_{i+1(i+2)}} E_{II}^+(x) dx + \int_{\alpha_{(n-1)n}}^1 E_{III}(x) dx$$

and by inversion:



$$\begin{aligned} \rho_n = & \frac{n\pi - \frac{\pi}{2}}{\int_0^1 \sqrt{r_+(x)} dx} - \frac{1}{n\pi} \left[\frac{[4(\nu_1 - 1)^2 - 1] + [4(\nu_2 - 1)^2 - 1]}{4 \int_{x_1}^{x_2} \sqrt{r(t)} dt} + \frac{[4(\nu_2 - 1)^2 - 1] + [4(\nu_3 - 1)^2 - 1]}{4 \int_{x_2}^{x_3} \sqrt{r(t)} dt} \right. \\ & \left. + \dots + \frac{[4(\nu_{n-1} - 1)^2 - 1] + [4(\nu_n - 1)^2 - 1]}{4 \int_{x_{n-1}}^{x_n} \sqrt{r(t)} dt} - \frac{1}{2} P(x_1, x_2, x_3, \dots, x_n) \right] + O\left(\frac{1}{n^2}\right). \end{aligned} \quad (12)$$

REMARK. Note that the reader can obtain asymptotic distribution of eigenvalues in different types of (TP) by consideration of combination of the above cases.

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Email:dastmalchi@iaut.ac.ir