Robust Observer Design for T-S Systems with Nonlinear Consequent Parts

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***Abstract*—This paper considers the design of robust observers for a class of continuous-time nonlinear systems presented by Takagi-Sugeno (T-S) model with nonlinear subsystems and unmeasurable premise variables. The proposed T-S structure reduces the number of rules in the Sugeno model by using local nonlinear rules. Moreover, it can represent larger class of nonlinear systems as compared to the measurable premise variable case. The proposed observer guarantees exponential convergence of state estimation error by Lyapunov stability analysis and linear matrix inequality (LMI) formulation in the presence of both parametric and non-parametric uncertainties. Numerical examples illustrate effectiveness of the proposed method.**

Keywords-Robust fuzzy observer; Nonlinear Local Model; Unmeasurable premise variable.

I. INTRODUCTION

TAKAGI-SUGENO (T-S) fuzzy model is a well-known tool for nonlinear system modeling with increasing interest in recent years. Because T-S model is a universal approximator, it can model any smooth nonlinear system with any degree of accuracy [1]. Furthermore, the local linear subsystems of this model allow one to use powerful linear systems tools, such as Linear Matrix Inequalities (LMIs), to analyze and synthesize T-S fuzzy systems. However, as complexity of the system increases, the number of rules in the model and hence, the number and dimensions of LMIs increases and becomes harder to solve. One possible solution is to reduce the accuracy in the model, which decreases the model complexity; however, the convergence of fuzzy controller or observer is not guaranteed in this case. To solve this problem, one may use nonlinear local subsystems for the T-S model. This will decrease the number of rules while increasing the model accuracy.

A very simple form of these nonlinear Sugeno model is used in [2] that has used a linear form for the consequence part plus a sinusoidal term. A more advanced work is performed by Dong in [3] and [4], where he used sector-bounded functions in the subsystems. In [5] and [6], Tanaka et al. have proposed the T-S model with polynomial subsystems. For stability analysis, they have used Sum of Squares (SOS) approach, which was the first use of SOS instead of LMI in fuzzy systems analysis. In [7] and [8], Sala represents a similar form of Sugeno model and used the Taylor series expansion of the system for construction of the polynomial subsystems. He states that the nonlinear consequent part in the T-S model not only reduces the number of rules but also reduces the conservativeness in the controller design. Design of fuzzy observers for such systems is briefly discussed here.

Fuzzy observers were first introduced by Tanaka and Sano in [9] in 1994 and ever since have been an active research issue. Different types of fuzzy observers have been discussed in literatures. The separation property of fuzzy observer and controller was first discussed by Ma et al. in [10] and more completely by Yan and Sun in [11]. When uncertainty exists in the model, the separation property cannot be proved in general. Later, in this regard, robust fuzzy observers were discussed in different papers such as [12] and [13].

For estimating the states of T–S systems, two cases for the premise variable can be distinguished. First, the premise variable vector does not depend on the estimated states; and second, when the premise variable depends on some of the states to be estimated. The latter structure can represent a larger class of non-linear systems. Unfortunately, the developed methods for fuzzy observer design with measured premise variables are not directly applicable for the systems with premise variables as a function of states [14]. Most works in literatures are based on the first case. The second case was first discussed in [15] in 2001. Many recent works are also working on obtaining less conservative conditions for this case [14] and [16]. For T-S systems with nonlinear consequent and unmeasurable premise variables, when no uncertainty exist in the model state observer is designed in [17]. In this paper, robust state observer for T-S fuzzy systems with nonlinear consequent part is designed when the premise variables depend on the estimated states. Both parametric and non-parametric uncertainties are considered in the model.

The paper is organized as follows: In Section 2 the nonlinear Sugeno model is described. The proposed observer and its convergence proof for systems with non-parametric uncertainty are presented in Section 3 and with parametric uncertainty in Section 4. In Section 5, two numerical examples are given to show effectiveness of the proposed method.

II. SUGENO MODEL WITH NONLINEAR SUBSYSTEMS

Consider a class of nonlinear system described by

$$\begin{aligned}\dot{x}(t) &= f_a\left(x(t)\right)+f_b\left(x(t)\right)\varphi\left(x(t)\right)+g\left(x(t)\right)u(t)\\ y(t) &= f_{ya}\left(x(t)\right)+f_{yb}\left(x(t)\right)\varphi\left(x(t)\right)\end{aligned} \tag{1}$$

where $x(t)$ is the state, $u(t)$ is the control input, $y(t)$ is the measurable output, $f_n(x(t))$ $n\in\{a,b,ya,yb\}$ are nonlinear functions and $\varphi(x(t))$ is a vector of Lipschitz nonlinear functions, satisfying Lipschitz condition

$$\left\|\varphi_i(x(t))-\varphi_i(\hat{x}(t))\right\| \le \theta_i \left\|\mathbf{R}_i\left(x(t)-\hat{x}(t)\right)\right\| \quad 1 \le i \le s \tag{2}$$

where θ_i is a Lipschitz constant, R_i is a constant matrix with appropriate dimensions, and s is the number of nonlinear functions.

The system (1) can be represented by a T-S fuzzy system with local nonlinear models as follows:

$$\begin{aligned}
&\textit{Plant Rule } i: \\
&\text{IF } z_1(t) \text{ is } \mu_{i1}(z), \ldots, \text{ and } z_p(t) \text{ is } \mu_{ip}(z), \quad \text{THEN} \\
&\dot{x}(t)=\mathbf{A}_i x(t)+\mathbf{G}_{xi}\varphi\left(x(t)\right)+\mathbf{B}_i u(t)+\mathbf{D}_{1i}\omega(t) \\
&y(t)=\mathbf{C}_i x(t)+\mathbf{G}_{yi}\varphi\left(x(t)\right)+\mathbf{D}_{2i}\omega(t)
\end{aligned} \tag{3}$$

where $i=1,\ldots,r$ is the number of rules, $z_1(t),\ldots,z_p(t)$ are the premise variables, and μ_{ij} denote the fuzzy sets.

In this case, the whole fuzzy system can be represented as

$$\begin{aligned}
\dot{x}(t)&=\sum_{i=1}^{r}\omega_i(z)\left[\mathbf{A}_i x(t)+\mathbf{G}_{xi}\varphi\left(x(t)\right)+\mathbf{B}_i u(t)+\mathbf{D}_{1i}\omega(t)\right] \\
y(t)&=\sum_{i=1}^{r}\omega_i(z)\left[\mathbf{C}_i x(t)+\mathbf{G}_{yi}\varphi\left(x(t)\right)+\mathbf{D}_{2i}\omega(t)\right]
\end{aligned} \tag{4}$$

where

$$\omega_i(z)=\frac{h_i(z)}{\sum_{k=1}^{r}h_k(z)},\ \sum_{i=1}^{r}\omega_i(z)=1,\ h_i(z)=\prod_{j=1}^{p}\mu_{ij}(z)$$

III. Observer Design

The observer used in this paper is a Luenberger type observer, given as follows:

$$\begin{aligned}
&\textit{Observer Rule } i: \\
&\text{IF } z_1(t) \text{ is } \mu_{i1}(z), \ldots, \text{ and } z_p(t) \text{ is } \mu_{ip}(z), \quad \text{THEN} \\
&\dot{\hat{x}}(t)=\mathbf{A}_i\hat{x}(t)+\mathbf{G}_{xi}\varphi\left(\hat{x}(t)\right)+\mathbf{B}_i u(t)+\mathbf{L}_i\left[\hat{y}(t)-y(t)\right] \\
&\hat{y}(t)=\mathbf{C}_i\hat{x}(t)+\mathbf{G}_{yi}\varphi\left(\hat{x}(t)\right)
\end{aligned} \tag{5}$$

For the analysis of the error convergence, two cases are distinguished: 1) the scheduling vector z does not depend on the estimated states, and 2) the scheduling vector z depends on some of the estimated states, which is the subject of this paper. In this case, the observer (5) becomes

$$\begin{aligned}
\dot{\hat{x}}(t)&=\sum_{i=1}^{r}\omega_i(\hat{z})\left\{\mathbf{A}_i\hat{x}(t)+\mathbf{G}_{xi}\varphi\left(\hat{x}(t)\right)+\mathbf{B}_i u(t)+\mathbf{L}_i\left[\hat{y}(t)-y(t)\right]\right\} \\
\hat{y}(t)&=\sum_{i=1}^{r}\omega_i(\hat{z})\left[\mathbf{C}_i\hat{x}(t)+\mathbf{G}_{yi}\varphi\left(\hat{x}(t)\right)\right]
\end{aligned} \tag{6}$$

For the sake of notation, the following definitions will be used:

$$\begin{aligned}
\mathbf{X}_{\hat{z}} &:= \sum_{i=1}^{r}\omega_i(\hat{z})\mathbf{X}_i \\
\mathbf{X}_{z} &:= \sum_{i=1}^{r}\omega_i(z)\mathbf{X}_i
\end{aligned} \tag{7}$$

for $\mathbf{X}\in\left[\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{G}_x,\mathbf{G}_y,\mathbf{L}\right]$. Hence, the error dynamic can be rewritten as

$$\begin{aligned}
\dot{e}(t)=&\left(\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}}\right)e(t)+\left(\mathbf{G}_{x\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}}\right)\varphi_e\left(x(t)\right) \\
&+\left(\mathbf{D}_{1\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)\omega(t)+\mathbf{\Delta}(t)
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\mathbf{\Delta}(t)=&\sum_{i=1}^{r}\left(\omega_i(\hat{z})-\omega_i(z)\right)\left((\mathbf{A}_i+\mathbf{L}_{\hat{z}}\mathbf{C}_i)x(t)+\right. \\
&\left.(\mathbf{G}_{xi}+\mathbf{L}_{\hat{z}}\mathbf{G}_{yi})\varphi\left(x(t)\right)+\left(\mathbf{D}_{1i}+\mathbf{L}_{\hat{z}}\mathbf{D}_{2i}\right)\omega(t)+\mathbf{B}_i u(t)\right)
\end{aligned} \tag{9}$$

and

$$\varphi_e\left(x(t)\right)=\varphi\left(\hat{x}(t)\right)-\varphi\left(x(t)\right) \tag{10}$$

A. Fuzzy Observer Analysis

In this section, the conditions for asymptotically convergence of the observer states in (5) to the system states in (4) will be given. The following lemmas are used in this paper.

Lemma 1 [3]: If the following conditions hold:

$$\mathbf{M}_{ii}<0 \quad 1<i<r$$

$$\frac{1}{r-1}\mathbf{M}_{ii}+\frac{1}{2}(\mathbf{M}_{ij}+\mathbf{M}_{ji})<0 \quad 1<i\neq j<r$$

Then, the following inequality holds:

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\alpha_i\alpha_j\mathbf{M}_{ij}<0$$

where α_i $(1\le i\le r)$ satisfies $0\le\alpha_i\le 1, \sum_{i=1}^{r}\alpha_i=1$.

Lemma 2 [18]: For any positive-definite matrix $\mathbf{\Pi}$ with appropriate dimensions, the following property holds:

$$\mathbf{X}^T\mathbf{Y}+\mathbf{Y}^T\mathbf{X}\le\mathbf{X}^T\mathbf{\Pi}\mathbf{X}+\mathbf{Y}^T\mathbf{\Pi}^{-1}\mathbf{Y} \qquad \mathbf{\Pi}>0$$

In the following theorem, sufficient stability conditions for error dynamic (8) will be given.

Theorem 1: The error dynamic (8) is asymptotically stable and with an H_∞ performance bound $\gamma>0$ if there exist $\mathrm{P}=\mathrm{P}^T>0$, Y_i $1\le i\le r,$ $\eta,\eta_1>0$, $\Lambda=\mathrm{diag}[\lambda_1\ldots\lambda_s]_{s\times s}$, $\lambda_i>0$ such that

$$\left\|\mathbf{\Delta}(t)\right\|\le\eta_1\left\|e(t)\right\| \tag{11}$$

$$\begin{aligned}
&\mathbf{\Xi}_{ii}<0 \quad 1\le i\le r, \\
&\frac{1}{r-1}\mathbf{\Xi}_{ii}+\frac{1}{2}(\mathbf{\Xi}_{ij}+\mathbf{\Xi}_{ji})<0 \quad 1\le i\neq j\le r
\end{aligned} \tag{12}$$

where $\mathbf{\Delta}(t)$ is defined in (9) and

$$\mathbf{\Xi}_{ij}=\begin{bmatrix} He\left(\mathbf{A}_i^T\mathbf{P}+\mathbf{C}_i^T\mathbf{Y}_j^T\right)+\mathbf{R}^T\theta\mathbf{\Lambda}\mathbf{R}+\eta\mathbf{I}+\gamma^2\mathbf{I} & * & * & * \\ \mathbf{G}_{xi}^T\mathbf{P}+\mathbf{G}_i\mathbf{Y}_j^T & -\mathbf{\Lambda} & * & * \\ \mathbf{P} & 0 & -\mathbf{I} & * \\ \mathbf{D}_{1i}^T\mathbf{P}+\mathbf{D}_{2i}^T\mathbf{Y}_j^T & 0 & 0 & -\mathbf{I} \end{bmatrix}$$

in which $He(\mathbf{X}) = \mathbf{X} + \mathbf{X}^T$, $\eta = \eta_1^2$, $\mathbf{R} = [\mathbf{R}_1^T \quad \mathbf{R}_2^T \quad \dots \quad \mathbf{R}_s^T]$, $\theta = \text{diag}[\theta_1^2 I_1 \quad \dots \quad \theta_s^2 I_s]$ and I_i is an identity matrix, with n_i the number of rows of R_i. Then, the observerb gains are $\mathbf{L}_i = \mathbf{P}^{-1}\mathbf{Y}_i \quad 1 \le i \le r$.

Proof: Based on Lemma 1, condition (12) results in

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i(\hat{z})\omega_j(\hat{z})\mathbf{\Xi}_{ij} < 0$$

This means that

$$\begin{bmatrix} He\left(\mathbf{A}_{\hat{z}}^T\mathbf{P}+\mathbf{C}_{\hat{z}}^T\mathbf{Y}_{\hat{z}}^T\right)+\theta^T\mathbf{R}\Lambda\theta+\eta\mathbf{I}+\gamma^2\mathbf{I} & * & * & * \\ \mathbf{G}_{x\hat{z}}^T\mathbf{P}+\mathbf{G}_{\hat{z}}\mathbf{Y}_{\hat{z}}^T & -\mathbf{\Lambda} & * & * \\ \mathbf{P} & 0 & -\mathbf{I} & * \\ \mathbf{D}_{1\hat{z}}^T\mathbf{P}+\mathbf{D}_{2\hat{z}}^T\mathbf{Y}_{\hat{z}}^T & 0 & 0 & -\mathbf{I} \end{bmatrix} < 0 \quad (13)$$

Pre- and post-multiplying (13) by $[e^T(t) \quad \varphi_e^T(x(t)) \quad \Delta(t) \quad \omega(t)]$ and its transpose, it yields

$$\begin{aligned} &\left[e^T(t)(\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})^T+\varphi_e^T(x(t))(\mathbf{G}_{x\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\right]\mathbf{P}e(t) \\ &+e^T(t)\mathbf{P}\left[(\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})e(t)+(\mathbf{G}_{x\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\varphi_e(x(t))\right]+ \\ &e^T(t)\mathbf{R}^T\theta\mathbf{\Lambda}\mathbf{R}\,\mathrm{e}(t)+\eta e^T(t)\mathrm{e}(t)-\varphi_e^T(t)\mathbf{\Lambda}\varphi_e(t)-\Delta^T(t)\Delta(t) \\ &+\Delta^T(t)\mathbf{P}e(t)+e^T(t)\mathbf{P}\Delta(t)+\gamma e^T(t)\mathrm{e}(t)-\omega^T(t)\omega(t) \\ &\omega(t)^T\left(\mathbf{D}_{1\hat{z}}+L_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)^T\mathbf{P}e(t)+e^T(t)\mathbf{P}\left(\mathbf{D}_{1\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)\omega(t)<0 \end{aligned} \quad (14)$$

Since $\varphi(x(t))$ satisfies the Lipschitz condition (2) and $\mathbf{\Lambda} = \text{diag}[\lambda_1 \dots \lambda_s]_{s\times s} \quad \lambda_i > 0$, it results that

$$e^T(t)\mathbf{R}^T\theta\mathbf{\Lambda}\mathbf{R}\,\mathrm{e}(t)-\varphi_e^T(t)\mathbf{\Lambda}\varphi_e(t)\ge 0 \quad (15)$$

Also based on (11)

$$\eta e^T(t)\mathrm{e}(t)-\Delta^T(t)\Delta(t)\ge 0 \quad (16)$$

Combining (15), (16) and (14) gives

$$\begin{aligned} &\left[e^T(t)(\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})^T+\varphi_e^T(x(t))(\mathbf{G}_{x\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\right]\mathbf{P}e(t) \\ &+e^T(t)\mathbf{P}\left[(\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})e(t)+(\mathbf{G}_{x\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\varphi_e(x(t))\right] \\ &+\Delta^T(t)\mathbf{P}e(t)+e^T(t)\mathbf{P}\Delta(t) \\ &+He\left(\omega(t)^T\left(\mathbf{D}_{1\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)^T\mathbf{P}e(t)\right) \\ &+\gamma^2 e^T(t)\mathrm{e}(t)-\omega^T(t)\omega(t)<0 \end{aligned} \quad (17)$$

For error dynamic (8), selecting the Lyapunov function as $V(t) = e^T(t)Pe(t)$, it yields

$$\begin{aligned} \dot{V}(t)=&\left[e^T(t)(\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})^T+\varphi_e^T(x(t))(\mathbf{G}_{x\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\right]\mathbf{P}e(t) \\ &+e^T(t)\mathbf{P}\left[(\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})e(t)+(\mathbf{G}_{x\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\varphi_e(x(t))\right] \\ &+\omega^T(t)\left(\mathbf{D}_{1\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)^T\mathbf{P}e(t)+e^T(t)\mathbf{P}\left(\mathbf{D}_{1\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)\omega(t) \\ &+\Delta^T(t)\mathbf{P}e(t)+e^T(t)\mathbf{P}\Delta(t) \end{aligned} \quad (18)$$

Comparing (18) and (17) results that

$$\dot{V}(t)+\gamma^2 e(t)e^T(t)-\omega^T(t)\omega(t)<0 \quad (19)$$

which implies that the error dynamic is asymptotically stable and $\|e(t)\| < \gamma\|\omega(t)\|$. □

Remark 1: In order to satisfy (11), the maximum value for η should be found. Hence, the problem of observer design can be stated as the generalized eigenvalue problem (GEVP) as

Maximize η
subject to (12)

Formulating as a GEVP, the value of η can be determined by the LMI solvers, and there is no need to determine this parameter in advance.

Remark 2: condition (11) is a mild condition which is satisfied if $\omega_i(z)$ is differentiable w.r.t. x and have a bounded first derivative for almost all x. This is satisfied by most membership functions in practice.

IV. Parametric Uncertainty Case

Now suppose there are some uncertainties in the model parameters as follows:

Plant Rule i :

$$\begin{aligned} &\text{IF } z_1(t) \text{ is } \mu_{i1}(z), \dots, \text{ and } z_p(t) \text{ is } \mu_{ip}(z), \text{ THEN} \\ &\dot{x}(t)=(\mathbf{A}_i+\Delta\mathbf{A}_i)x(t)+\mathbf{G}_{xi}\varphi(x(t))+\mathbf{B}_i u(t)+\mathbf{D}_{1i}\omega(t) \\ &y(t)=(\mathbf{C}_i+\Delta\mathbf{C}_i)x(t)+\mathbf{G}_{yi}\varphi(x(t))+\mathbf{D}_{2i}\omega(t) \end{aligned} \quad (20)$$

where

$$\Delta\mathbf{A}_i=\mathbf{M}_{1i}\mathbf{F}\mathbf{N}_1,\ \Delta\mathbf{C}_i=\mathbf{M}_{2i}\mathbf{F}\mathbf{N}_2,\ \mathbf{F}^T\mathbf{F}<1 \quad (21)$$

From now on, it is assumed that the nonlinear functions are sector bounded. I.e., for any $\mathbf{\Lambda}_1 > 0$

$$\varphi(x(t))^T\mathbf{E}\mathbf{\Lambda}_1 x(t)-\varphi(x(t))^T\mathbf{\Lambda}_1\varphi(x(t))\ge 0 \quad (22)$$

where $\mathbf{E}$ is a matrix with proper dimensions. For such a model, an observer of the form (5) is considered. Then, the error dynamic would be

$$\begin{aligned} \dot{e}(t)=&\left(\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}}\right)e(t)+\left(\mathbf{G}_{x\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}}\right)\varphi_e(x(t)) \\ &+\left(\Delta\mathbf{A}_{\hat{z}}+\mathbf{L}_{\hat{z}}\Delta\mathbf{C}_{\hat{z}}\right)x(t)+\left(\mathbf{D}_{1\hat{z}}+\mathbf{L}_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)\omega(t)+\mathbf{\Delta}_1(t) \end{aligned} \quad (23)$$

Now let rewrite the system equation as

$$\begin{aligned} \dot{x}(t)=&(\mathbf{A}_{\hat{z}}+\Delta\mathbf{A}_{\hat{z}})x(t)+\mathbf{G}_{x\hat{z}}\varphi(x(t)) \\ &+\mathbf{B}_{\hat{z}}u(t)+\mathbf{D}_{1\hat{z}}\omega(t)+\mathbf{\Delta}_2(t) \end{aligned}$$

where

$$\begin{aligned} &\mathbf{\Delta}_1(t)=\sum_{i=1}^{r}\left(\omega_i(\hat{z})-\omega_i(z)\right)\times \\ &\begin{bmatrix}(\mathbf{A}_i+\Delta\mathbf{A}_i+\mathbf{L}_{\hat{z}}\mathbf{C}_i+\mathbf{L}_{\hat{z}}\Delta\mathbf{C}_i)x(t)+\mathbf{B}_i u(t) \\ +(\mathbf{G}_{xi}+\mathbf{L}_{\hat{z}}\mathbf{G}_{yi})\varphi(x(t))+\left(\mathbf{D}_{1i}+\mathbf{L}_{\hat{z}}\mathbf{D}_{2i}\right)\omega(t)\end{bmatrix} \\ &\mathbf{\Delta}_2(t)=\sum_{i=1}^{r}\left(\omega_i(\hat{z})-\omega_i(z)\right)\times \\ &\left[(\mathbf{A}_i+\Delta\mathbf{A}_i)x(t)+\mathbf{B}_i u(t)+\mathbf{G}_{xi}\varphi(x(t))+\mathbf{D}_{1i}\omega(t)\right] \end{aligned} \quad (24)$$

In the following theorem, sufficient stability conditions for error dynamic (8) will be given.

Theorem 2: The error dynamic (23) is asymptotically stable and with an H_∞ performance bound $\gamma > 0$ if there exist $\mathrm{P}_1 = \mathrm{P}_1^T > 0$, $\mathrm{P}_2 = \mathrm{P}_2^T > 0$, $Y_i \quad 1 \le i \le r$, $\eta_1, \eta_2 > 0$, $\Lambda = \mathrm{diag}[\lambda_1 \ldots \lambda_s]_{s\times s}$, $\lambda_i > 0$ such that

$$\left\| \Delta_i(t) \right\| \le \eta_i \left\| e(t) \right\| \quad i = 1,2 \tag{25}$$

$$\Xi_{ii} < 0 \quad 1 \le i \le r,$$
$$\frac{1}{r-1}\Xi_{ii} + \frac{1}{2}(\Xi_{ij} + \Xi_{ji}) < 0 \quad 1 \le i \ne j \le r \tag{26}$$

where $\Delta_i(t)$ is defined in (24) and

$$\Xi_{ij} = \begin{bmatrix} S_1 & * & * & * & * & * & * \\ 0 & S_2 & * & * & * & * & * \\ S_3 & 0 & -\mathbf{\Lambda}_1 & * & * & * & * \\ 0 & S_4 & 0 & -\mathbf{\Lambda}_2 & * & * & * \\ S_5 & S_6 & 0 & 0 & -\gamma^2\mathbf{I} & * & * \\ S_7 & 0 & 0 & 0 & 0 & -\mathbf{I} & * \\ 0 & S_8 & 0 & 0 & 0 & 0 & -\mathbf{I} \end{bmatrix} \tag{27}$$

$$S_1 = \begin{bmatrix} He\left(\mathbf{A}_i^T\mathbf{P}_1 + \mathbf{C}_i^T\mathbf{Y}_j^T\right) + \mathbf{R}^T\theta\mathbf{\Lambda}\mathbf{R} + \eta_1^2\mathbf{I} + \gamma^2\mathbf{I} & * & * \\ \mathbf{M}_{1i}^T\mathbf{P}_1 & -\mathbf{I} & * \\ \mathbf{M}_{2i}^T\mathbf{Y}_j & 0 & -\mathbf{I} \end{bmatrix}$$

$$S_2 = \begin{bmatrix} He\left(\mathbf{A}_i^T\mathbf{P}_2\right) + \left(k^2 + \eta_2^2\right)\mathbf{I} + 2\mathbf{N}_1^T\mathbf{N}_1 + \mathbf{N}_2^T\mathbf{N}_2 & * & * \\ \mathbf{M}_{1i}^T\mathbf{P}_2 & -\mathbf{I} & * \\ \mathbf{B}_i^T\mathbf{P}_2 & 0 & -\mathbf{I} \end{bmatrix}$$

$$S_3 = \begin{bmatrix}\mathbf{G}_{xi}^T\mathbf{P}_1 + \mathbf{G}_{yi}^T\mathbf{Y}_j^T & 0 & 0\end{bmatrix},\ S_4 = \begin{bmatrix}\mathbf{G}_{xi}^T\mathbf{P}_2 + \mathbf{\Lambda}\mathbf{E} & 0 & 0\end{bmatrix}$$

$$S_5 = \begin{bmatrix}\mathbf{D}_{1i}^T\mathbf{P}_1 + \mathbf{D}_{2i}^T\mathbf{Y}_j^T & 0 & 0\end{bmatrix},\ S_6 = \begin{bmatrix}\mathbf{D}_{1i}^T\mathbf{P}_2 & 0 & 0\end{bmatrix}$$

$$S_7 = \begin{bmatrix}\mathbf{P}_1 & 0 & 0\end{bmatrix},\ S_8 = \begin{bmatrix}\mathbf{P}_2 & 0 & 0\end{bmatrix},\ \mathbf{Y}_i = \mathbf{P}\mathbf{L}_i$$

Proof: Based on Lemma 1, condition (24) results in

$$\sum_{i=1}^{r}\sum_{j=1}^{r}\omega_i(\hat{z})\omega_j(\hat{z})\Xi_{ij} < 0 \tag{28}$$

For the error dynamic (23), consider the following Lyapunov function:

$$V(t) = V_1(t) + V_2(t) = e^T(t)\mathbf{P}_1 e(t) + x(t)^T\mathbf{P}_2 x(t)$$

Then, it yields

$$\begin{aligned}\dot{V}_1(t) = &\left[e^T(t)(\mathbf{A}_{\hat{z}} + \mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})^T + \varphi_e^T(x(t))(\mathbf{G}_{x\hat{z}} + \mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\right]\mathbf{P}_1 e(t) \\ &+ e^T(t)\mathbf{P}_1\left[(\mathbf{A}_{\hat{z}} + \mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})e(t) + (\mathbf{G}_{x\hat{z}} + \mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\varphi_e(x(t))\right] \\ &+ 2e^T(t)\mathbf{P}_1\left[\mathbf{M}_{1\hat{z}}\mathbf{F}\mathbf{N}_1 + \mathbf{L}_{\hat{z}}\mathbf{M}_{2\hat{z}}\mathbf{F}\mathbf{N}_2\right]x(t) \\ &+ He\left(\omega^T(t)\left(\mathbf{D}_{1\hat{z}} + \mathbf{L}_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)^T\mathbf{P}_1 e(t)\right) \\ &+ \Delta_1^T(t)\mathbf{P}_1 e(t) + e^T(t)\mathbf{P}_1\Delta_1(t)\end{aligned}$$

$$\begin{aligned}\dot{V}_2(t) = &\, x^T(t)\left[\mathbf{A}_{\hat{z}}^T\mathbf{P}_2 + \mathbf{P}_2\mathbf{A}_{\hat{z}}\right]x(t) + 2x^T(t)\mathbf{P}_2\mathbf{M}_{1\hat{z}}\mathbf{F}\mathbf{N}_1 x(t) \\ &+ 2x^T(t)\mathbf{P}_2\left[\mathbf{G}_{x\hat{z}}\varphi(x(t)) + \mathbf{B}_{\hat{z}}u(t) + \mathbf{D}_{1\hat{z}}\omega(t)\right] \\ &+ \Delta_2^T(t)\mathbf{P}_2 x(t) + x^T(t)\mathbf{P}_2\Delta_2(t)\end{aligned}$$

Based on (21) and Lemma 2, it yields

$$\begin{aligned}&\dot{V}(t) \le \\ &x^T(t)\left[He\left(\mathbf{A}_{\hat{z}}^T\mathbf{P}_2\right) + \mathbf{P}_2\mathbf{M}_{2\hat{z}}\mathbf{M}_{2\hat{z}}^T\mathbf{P}_2 + \mathbf{N}_1^T\mathbf{N}_1 + \mathbf{N}_2^T\mathbf{N}_2\right]x(t) \\ &+ e^T(t)\left[He\left((\mathbf{A}_{\hat{z}} + \mathbf{L}_{\hat{z}}\mathbf{C}_{\hat{z}})^T\mathbf{P}_1\right) + \mathbf{P}_1\mathbf{L}_{\hat{z}}\mathbf{M}_{1\hat{z}}\mathbf{M}_{1\hat{z}}^T\mathbf{L}_{\hat{z}}\mathbf{P}_1\right]e(t) \\ &+ 2e^T(t)\mathbf{P}_1(\mathbf{G}_{x\hat{z}} + \mathbf{L}_{\hat{z}}\mathbf{G}_{y\hat{z}})\varphi_e(x(t)) \\ &+ 2x^T(t)\mathbf{P}_2\left[\mathbf{G}_{x\hat{z}}\varphi(x(t)) + \mathbf{B}_{\hat{z}}u(t) + \mathbf{D}_{1\hat{z}}\omega(t)\right] \\ &+ 2e^T(t)\mathbf{P}_1\left(\mathbf{D}_{1\hat{z}} + \mathbf{L}_{\hat{z}}\mathbf{D}_{2\hat{z}}\right)\omega(t) \\ &+ 2e^T(t)\mathbf{P}_1\Delta_1(t) + 2x^T(t)\mathbf{P}_2\Delta_2(t)\end{aligned} \tag{29}$$

Suppose $\left\|u(t)\right\| < k\left\|x(t)\right\|$. Then, based on (29), (15), (16) and (21) and by Pre- and post-multiplying (27) after the use of Schur-complement Lemma on S_1 and S_2 by $\begin{bmatrix}e^T & x^T & \varphi_e^T & \varphi^T & \omega^T & \Delta_1 & \Delta_2\end{bmatrix}$ and its transpose it yields $\dot{\mathbf{V}}(t) + \gamma e(t)e^T(t) - \gamma^{-1}\omega^T(t)\omega(t) \le 0$ which results $\left\|e(t)\right\| < \gamma\left\|\omega(t)\right\| \quad t \to \infty$ so the proof is complete.

V. Simulation Examples

A. Example 1

Consider the system shown in Fig. 1, which represents a Translational Oscillator with an eccentric Rotational Actuator (TORA) [19]. The nonlinear coupling between the rotational motion of the actuator and the translational motion of the oscillator provides the mechanism for control. Let x_1 and x_2 denote the translational position and velocity of the cart and x_3 and x_4 denote the angular position and velocity of the rotational mass. Then, the system dynamics can be described by the equation

$$\dot{x} = \begin{bmatrix} x_2 \\ \dfrac{-x_1 + \varepsilon x_4^2\sin x_3}{1-\varepsilon^2\cos^2 x_3} \\ x_4 \\ \dfrac{\varepsilon\cos x_3\left(x_1 - \varepsilon x_4^2\sin x_3\right)}{1-\varepsilon^2\cos^2 x_3} \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{-\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3} \\ 0 \\ \dfrac{1}{1-\varepsilon^2\cos^2 x_3} \end{bmatrix}u + d_1$$

$$y = \begin{bmatrix}x_1 & x_3\end{bmatrix}^T + d_2$$

where

$$d_1 = \begin{bmatrix} 0 \\ \dfrac{1}{1-\varepsilon^2\cos^2 x_3} \\ 0 \\ \dfrac{-\varepsilon\cos x_3}{1-\varepsilon^2\cos^2 x_3} \end{bmatrix}\omega(t) \quad d_2 = \begin{bmatrix}0.1 \\ 0.2\end{bmatrix}\omega(t),\ \varepsilon = 0.05$$

in which $\omega(t)$ represents white noise. This system can be modeled as follows:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/(1-\varepsilon^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \varepsilon/(1-\varepsilon^2) & 0 & 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1/(1-\varepsilon^2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\varepsilon/(1-\varepsilon^2) & 0 & 0 & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -4\varepsilon \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 4\varepsilon \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B_1^T = \begin{bmatrix} 0 & -\varepsilon/(1-\varepsilon^2) & 0 & 1/(1-\varepsilon^2) \end{bmatrix}$$

$$B_2^T = B_4^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B_3^T = \begin{bmatrix} 0 & \varepsilon/(1-\varepsilon^2) & 0 & 1/(1-\varepsilon^2) \end{bmatrix}$$

$$G_{x1}^T = G_{x3}^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, G_{x2}^T = -G_{x4}^T = \begin{bmatrix} 0 & \varepsilon & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \varphi(x) = x_4^2 + 4x_4$$

Note that for this system, the premise variable x_3 is measured. Hence, the sixth row and the seventh column are omitted from Ξ_{ij} in (27). Moreover, the term $\eta_1^2\mathbf{I}$ in element (1,1) of S_1 and $\eta_2^2\mathbf{I}$ in element (1,1) of S_2 are omitted. In addition, note that the system is stabilized first and the observer is designed for the stable system. The following gains for the observer are obtained based on Theorem 2. Beside on the nonparametric disturbances define in the model of the system, parametric uncertainty is also added by considering $\varepsilon = 0.15$ for the observer design. Here $\gamma = 0.1$.

$$\mathrm{L}_1 = \begin{bmatrix} -1.5030 & 1.5972 \\ -28.3319 & 24.795 \\ 6.2794 & -8.6068 \\ 8.2667 & -10.9328 \end{bmatrix}, \mathrm{L}_2 = \begin{bmatrix} -1.4816 & 1.4442 \\ -28.7303 & 24.794 \\ 6.5644 & -8.7266 \\ 8.5149 & -11.1491 \end{bmatrix}$$

$$\mathrm{L}_3 = \begin{bmatrix} -1.4810 & 1.3014 \\ -30.5188 & 25.6819 \\ 7.3003 & -9.0517 \\ 9.3751 & -11.5306 \end{bmatrix}, \mathrm{L}_4 = \begin{bmatrix} -1.4770 & 1.4379 \\ -29.3607 & 25.540 \\ 6.8114 & -9.021 \\ 8.8464 & -11.545 \end{bmatrix}$$

Fig. 2 shows states and their estimation.

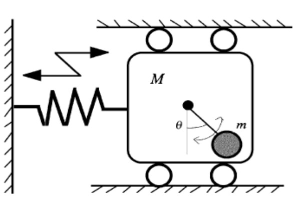

Figure 1. TORA system

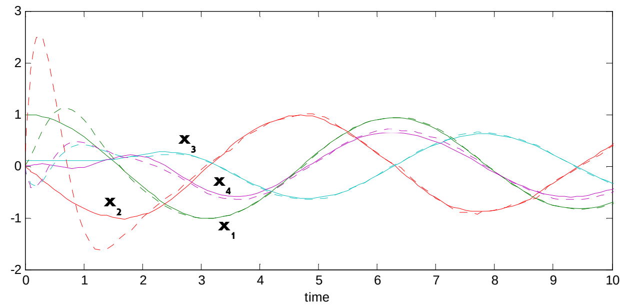

Figure 2. States (solid) and their estimation (doted) of TORA

B. Example 2

In this example, the ball-and-plate system [20] is considered. To design an observer for this system, it is assumed that the mutual interactions of both coordinates are negligible. Due to the symmetry of x and y directions, only the x direction is discussed here. The other direction has a similar behavior. The state-space form of the system in x direction can be given as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ b(x_1 x_4^2 - g\sin x_3) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u_x + d$$

where, $X = (x_1, x_2, x_3, x_4)^T = (x, \dot{x}, \theta_x, \dot{\theta}_x)^T$, $b = 0.7143$, and $g = 9.81 m/s^2$. Moreover, x is the position of the ball along the x-axis and θ_x is the angle of the plate measured from the x-axis. Assume that $x_1 x_4 \in [-d_1, d_1]$. This system can be modeled by a Sugeno model with at least 4 linear rules or by only two nonlinear rules based on (32) with $x_1\hat{x}_4$ as the premise variable and the following parameters:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & bd_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -bd_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_{x1} = G_{x2} = \begin{bmatrix} 0 \\ -bg \\ 0 \\ 0 \end{bmatrix}, B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, C_1 = C_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\omega_1 = (x_1\hat{x}_4 + d_1)/2d_1 \qquad \omega_2 = 1 - \omega_1$$

$$D_{1i} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_{2i} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}$$

In this example, $\varphi(x(t)) = \sin(x_3)$, $\theta = 1$, $d_1 = 2$, and $R = [0 \;\; 0 \;\; 1 \;\; 0]$. A bound limited white noise with power of 0.01 is added to the system as the external disturbances.

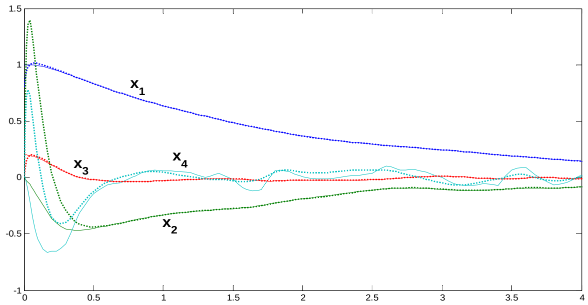

Figure 3. States in *x* direction (solid) and their estimation (doted) for the ball and plate system.

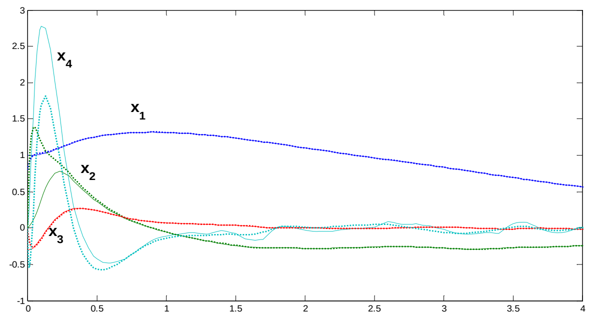

Figure 4. States in *y* direction (solid) and their estimation (doted) for the ball and plate system.

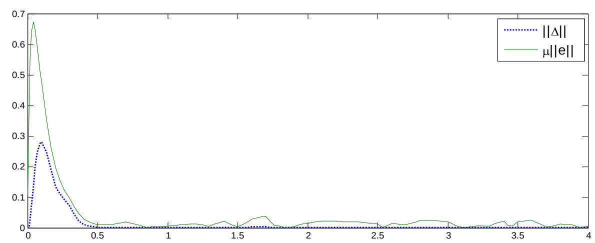

Figure 5. Satisfaction of (11) in example 2

Fig. 3 and Fig. 4 show the state estimation error of the observer designed based on Theorem 1. Note that the same observer is used for the *y* direction of the system. Observer gains are determined based on Theorem 1 and are equal to

$$L_1 = \begin{bmatrix} -41.7075 & -3.4428 \\ -355.4789 & -33.0751 \\ 5.3027 & -58.0486 \\ 39.1375 & -278.2707 \end{bmatrix} L_2 = \begin{bmatrix} -41.4013 & -3.6586 \\ -353.8834 & -36.5552 \\ 1.1029 & -58.1477 \\ -5.2313 & -279.2827 \end{bmatrix}$$

$\eta = 0.1584$, $\gamma = 0.5$

VI. Conclusion

A Sugeno-type fuzzy observer with nonlinear local subsystems and unmeasurable premise variables for a class of continuous-time non-linear systems with both parametric and non-parametric uncertainties is proposed in this paper. The use of non-linear consequent for the T-S system reduces the number of rules in the model. Moreover, by assuming unmeasurable premise variables, one can model larger class of non-linear systems. The estimation error convergence was shown using a quadratic Lyapunov function and LMI formulation. The future works include separation property check of the controller as well as the observer for this model.

VII. References

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