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Clean and pretty clean modules

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Abstract

In this talk we review some recent results about clean and pretty clean modules and their applications. Let R be a Noetherian ring and M an R-module. A chain $\mathcal{F}\colon (0) = M_0 \subset M_1 \subset \ldots \subset M_r = M$ of submodules of M is called a prime filtration of M, if for all $i = 1, \ldots, r$, there exists a prime ideal $P_i \in \operatorname{Spec}(R)$ such that $M_i/M_{i-1} \cong R/P_i$. If M is finitely generated such a prime filtration of M always exists. The set of prime ideals $P_1, \ldots, P_r$ which define the cyclic quotients of $\mathcal{F}$ will be denoted by $\operatorname{Supp}(\mathcal{F})$. It is easy to see that if $\mathcal{F}$ is a prime filtration of M, then $\operatorname{Ass}(M) \subset \operatorname{Supp}(\mathcal{F}) \subset \operatorname{Supp}(M)$.

Dress called the prime filtration $\mathcal{F}$ *clean* if $\operatorname{Supp}(\mathcal{F}) = \operatorname{Min}(M)$. The R-module M is called clean if it has a clean filtration.

Herzog and Popescu generalized this concept and they called a prime filtration $\mathcal{F}$ *pretty clean*, if for all $i < j$ which $P_i \subseteq P_j$ it follows that $P_i = P_j$. The R-module M is called pretty clean if it admits a pretty clean filtration. It follows that if $\mathcal{F}$ is a pretty clean filtration of M, then $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$. The converse of the above fact is not true. We call an R-module M *almost clean* if it admits a prime filtration $\mathcal{F}$ with $\operatorname{Supp}(\mathcal{F}) = \operatorname{Ass}(M)$.

Let K be a field and $S = K[x_1, \ldots, x_n]$ the polynomial ring in n variables. Let I be a monomial ideal in S. We say that I is (pretty) clean if S/I is (pretty) clean. In this talk we consider modules of the form S/I. Cleanness is the algebraic counterpart of shellability for simplicial complexes.

If Δ is a simplicial complex on vertex set $[n]$, there is a bijection between squarefree monomial ideals $I \subset (x_1, \ldots, x_n)^2$ and the simplicial complexes.

Let Δ be a simplicial complex with the set of facets $\{F_1, \ldots, F_t\}$. So an order $F_1, \ldots, F_t$ of the facets of Δ is called a shelling of Δ if the simplicial complex $\langle F_1, \ldots, F_{i-1}\rangle \cap \langle F_i\rangle$ is pure and $(\dim F_i - 1)$-dimensional for all $i = 2, \ldots, t$.

Theorem 1. The simplicial complex Δ is shellable if and only if I_Δ is clean.

We give an easy proof it by using induction. To each monomial ideal I one can attach a multicomplex $\Gamma(I)$ and vice versa. Then there is a bijection between monomial ideals and multicomplexes. Herzog and Popescu defined the shelling of multicomplexes and they proved the following:

Theorem 2. A multicomplex Γ is shellable if and only if $I(\Gamma)$ is pretty clean.

Then we use the polarization of monomial ideals to show that a monomial ideal I is pretty clean if and only if its polarization is clean. By using this fact we find some class of (pretty) clean monomial ideals.

Theorem 3. Let $I \subset S$ be a monomial ideal. Then S/I is (pretty) clean if I is almost complete intersection, Cohen-Macaulay of codimension 2, Gorenstien of Codimension 3, monomial ideal of forest type and monomial ideal with at most 3 generator.

We show that if a monomial I is pretty clean, then S/I is sequentially Cohen-Macaulay and $\operatorname{depth}(S/I) = \min\{\dim(S/P) : P \in \operatorname{Ass}(S/I)\}$. Also one can compute the regularity of S/I from the pretty clean filtration of S/I.

We also give a new characterization of (pretty) clean modules in terms of primary decomposition of their zero submodules. For this we need to characterize a prime filtration of a module in term of the primary decomposition of its zero submodule. Finally we give some applications of pretty clean modules.