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Different Analytical Methods For Strongly Nonlinear Oscillator With High Nonlinearity

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Abstract

Nonlinear oscillator models have been widely used in many areas of physics and engineering such as mechanical and structural problems. In this paper, different analytical methods are applied to the strongly nonlinear oscillator with high nonlinearity. Periodic solutions are analytically verified and consequently the relationship between the natural frequency and the initial amplitude is obtained in an analytical form. According to this article on comparative study of four analytical method to solve the equations of ordinary differential equation vibration beam is focused. The obtained results are compared with each other and exact solution. Accuracy and validity of the proposed techniques are then verified by comparing the numerical results obtained based on these methods and exact integration method. These methods are Coupled Homotopy Variational Formulation, Integral Iteration Method, Amplitude Frequency Formulation and Variational Approach. Finally, present analytical procedures is evaluated in comparison with numerical calculation methods.

Keywords: Analytical Methods, Nonlinear Oscillation, Periodic Solution, Natural Frequency.

Introduction

Most scientific problems in physics and engineering problems are inherently nonlinear. All these problems and phenomena are modeled by ordinary or partial differential equations. Except a limited number of these problems, most of them do not have any analytical solution. Nonlinear oscillator models have been widely used in many areas of physics and engineering and have significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion. The study of nonlinear oscillators has received considerable attention in recent years due to a variety of engineering applications.

Several approaches have been proposed so far dealing with the nonlinear oscillators. Some of these methods are including Variational Iteration Method (He et al, 2010), Coupled Homotopy Variational

Formulation (Akbarzade and Domiri-Ganji, 2010), Variational Approach (He, 2007), (Kaya et al, 2010), Amplitude Frequency Formulation (Zhang, 2009), (Ganji and Akbarzade, 2010), (He, 2009) and other classical methods (He, 2005), (Marinca and Herisanu, 2010).

According to this article on comparative study of four analytical method to solve the ordinary differential equation vibration of Euler-Bernoulli beam is focused. The obtained results of the method are then compared with the exact solution. Comparison of the result which is obtained by these methods with the obtained result by the exact solution reveals that these approaches are very effective and convenient.

The physical model of the problem, consider the Euler Bernoulli beam with length *l*, moment of inertia *I*, mass per unit length *m* and modulus of elasticity *E*, which is axially compressed by loading *P* as shown in Figure 1.

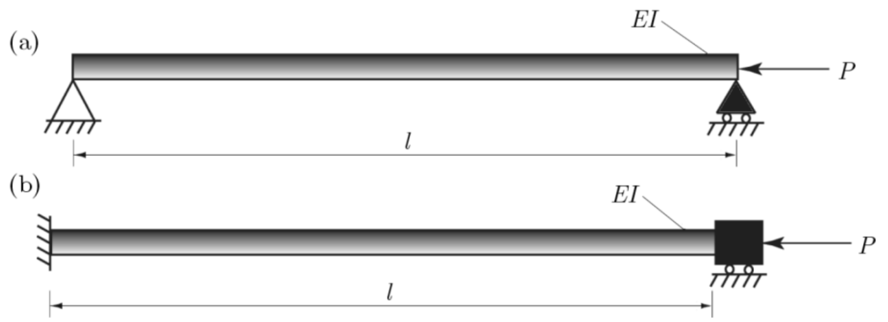

Fig. 1. Configuration of a uniform Euler-Bernoulli beam (a) simply supported beam, (b) clamped-clamped beam (Sedighi et al, 2013)

With evolution of technology, accurate comprehending of characteristics of beam vibrations is extremely important to researchers and engineers. The dynamic response of simply supported and clamped-clamped structures at large amplitudes of vibration can be encountered in many engineering applications. The problem of beam vibrations was recently investigated by many researchers with different boundary conditions and hypotheses. Nonlinear frequency of beams which are very important for the design of many engineering structures. The results presented in this paper exhibit that the analytical methods are very effective and convenient for nonlinear the beam vibration for which highly nonlinear governing equations (Sedighi et al, 2013).

From Sedighi et al, it is known that nonlinear oscillator equation to solve the ordinary differential equation vibration of Euler-Bernoulli beam is expressed by:

$$u'' + u + au^3 + bu^5 + cu^7 = 0 \qquad (1)$$

$$u(0) = A \quad , \quad u'(0) = 0 \qquad (2)$$

The general oscillator is rewritten in the following form (He, 2010):

$$u'' + f(u, u', u'') = 0 \qquad (3)$$

The application of the Coupled Homotopy Variational formulation (CHV)

The Coupled Method of Homotopy Perturbation Method (Ganji and Rajabi, 2006), (He, 2005), and Variational Formulation (He, 2007), (Kaya et al, 2010), couples the Homotopy Perturbation Method with the Variational Method. The method first constructs a homotopy equation, and then the solution is expanded into a series of p. As the zeroth order approximate solution is easy to be obtained, the second term is solved using the variational approach, where the frequency of the nonlinear oscillator can be obtained. The first-order solution is the best among all possible solutions when the trial solution is chosen in cosine or sine function. This technology is very much similar to Marinca's work where the unknown parameters are identified using least squares technology (Marinca and Herisanu, 2010).

The following homotopy can be constructed:

$$u'' + \omega^2 u + p\left[au^3 + bu^5 + cu^7 + (1-\omega^2)u\right] = 0,\ p \in [0,1] \quad (4)$$

When $p = 0$, Eq. (4) becomes the linearized equation, $u'' + \omega^2 u = 0$, when $p = 1$, it turns out to be the original one. Assume that the periodic solution to be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \ldots \quad (5)$$

Substituting Eq. (5) into Eq. (4), collecting terms of the same power of p, gives:

$$u_0'' + \omega^2 u_0 = 0,\ u_0(0) = A,\ u_0'(0) = 0 \quad (6)$$

And:

$$u_1'' + \omega^2 u_1 + au_0^{\,3} + bu_0^{\,5} + cu_0^{\,7} + (1-\omega^2)u_0 = 0,\ u_1(0) = 0,\ u_1'(0) = 0 \quad (7)$$

The solution of Eq. (6) is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads:

$$J(u_1) = \int_0^T \left\{ -\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^{\,2} + (1-\omega^2)u_0 u_1 + au_0^{\,3}u_1 + bu_0^{\,5}u_1 + cu_0^{\,7}u_1 \right\} dt,\ T = \frac{2\pi}{\omega} \quad (8)$$

To better illustrate the procedure, a simple trail function can be chosen:

$$u_1 = B(\cos\omega t - \frac{1}{3}\cos 5\omega t) \quad (9)$$

Substituting u_1 into the functional equation results in:

$$J(A,B,\omega) = \frac{B}{\omega}\left(A\pi + \frac{29}{48}bA^5\pi - A\omega^2\pi - \frac{4}{3}B\omega^2\pi + \frac{49}{96}cA^7\pi + \frac{3}{4}aA^3\pi \right) \quad (10)$$

Setting:

$$\frac{\partial J}{\partial B} = 0 \ \text{and}\ \frac{\partial J}{\partial \omega} = 0 \quad (11)$$

Solving the above equations, approximate frequency as a function of amplitude equals to:

$$\omega_{CHV} = \sqrt{1 + \frac{3}{4}aA^2 + \frac{29}{48}bA^4 + \frac{49}{96}cA^6} \quad (12)$$

For the case $a = 1$, $b = c = 0$, Eq. (1) turns to be the well-known Duffing equation which agrees exactly with our prediction. Similarly for the Coupled Homotopy Variational Formulation, the frequency ratio and the relative error are:

$$\frac{\omega_{CHV}}{\omega_{Exact}} = \frac{\sqrt{1 + \frac{3}{4}aA^2 + \frac{29}{48}bA^4 + \frac{49}{96}cA^6}}{2\pi\left(4\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 + \frac{1}{2}(1+\sin^2\theta)aA^2 + \frac{1}{3}(1+\sin^2\theta+\sin^4\theta)bA^4 + \frac{1}{4}(1+\sin^2\theta+\sin^4\theta+\sin^6\theta)cA^6}} \right)^{-1}} \quad (13)$$

And:

$$\lim_{A\to\infty} \frac{\omega_{CHV}}{\omega_{Exact}} = 1.0434 \quad (14)$$

The lowest order approximation given by (12) is actually within 4.3% of the exact frequency.

The application of the Integral Iteration Method (IIM)

In 2005, a simple but effective Integral Iteration Method was proposed to search for limit cycles or bifurcation curves of nonlinear equations by Ji-Huan He (He, 2005).

According to the method Eq. (3) can be rewritten in the following iteration form:

$$u_{n+1} = -u_n - f(u_n, u_n', u_n'') \tag{15}$$

Substituting a trail function into Eq. (3) and integrating twice, u_{n+1} can easily obtained.

Consider the nonlinear oscillator Eq. (1). For $n = 0$ the following iteration form is:

$$u_1 = -u_0 - au_0^3 - bu_0^5 - cu_0^7 \tag{16}$$

The simplest trail function is:

$$u_0 = A\cos\omega t \tag{17}$$

Substituting Eq. (15) into the functional Eq. (16) and integrating twice yields:

$$\begin{aligned} u_1 &= \frac{\cos(\omega t)}{\omega^2}\left(\frac{5}{8}bA^5 + \frac{35}{64}cA^7 + \frac{3}{4}aA^3 + A\right) + \\ &\frac{1}{\omega^2}\left(\frac{1}{3136}cA^7\cos(7\omega t) + \frac{7}{1600}cA^7\cos(5\omega t) + \frac{7}{192}cA^7\cos(3\omega t) + \frac{1}{36}aA^3\cos(3\omega t)\right. \\ &\left. + \frac{1}{400}bA^5\cos(5\omega t) + \frac{5}{144}bA^5\cos(3\omega t)\right) + Ct + B \end{aligned} \tag{18}$$

C and D are integral constants. The last two terms in (18) does not exhibit the periodic behavior, which is characteristic of oscillator equations, so they can be eliminated terms in the procedure (He, 2005).

By equating the coefficients of $\cos(\omega t)$ in u_0 and u_1 the approximate frequency can be obtained:

$$\omega_{IIM} = \sqrt{1 + \frac{3}{4}aA^2 + \frac{5}{8}bA^4 + \frac{35}{64}cA^6} \tag{19}$$

For the case $a = 1$, $b = c = 0$, Eq. (1) turns to be the well-known Duffing equation which agrees exactly with our prediction. Similarly for the Integral Iteration Method, the relative error is:

$$\lim_{A\to\infty} \frac{\omega_{IIM}}{\omega_{Exact}} = 1.0956 \tag{20}$$

The lowest order approximation given by (19) is actually within 9.5% of the exact frequency.

The application of the Amplitude Frequency Formulation (AFF)

To solve nonlinear problems, an Amplitude Frequency Formulation for nonlinear oscillators was proposed by He, which was deduced using an ancient Chinese mathematics method (Zhang, 2009), (Domiri-Ganji and Akbarzade, 2010), (He, 2009).

According to He's amplitude-frequency formulation, $u_1 = A\cos t$ and $u_2 = A\cos\omega t$ serve as the trial functions. Substituting u_1 and u_2 into Eq. (3), results in the following residuals:

$$R_1 = u_1''(t) + f\left(u_1(t), u_1'(t), u_1''(t)\right) \tag{21}$$

And

$$R_2 = u_2''(t) + f\left(u_2(t), u_2'(t), u_2''(t)\right) \tag{22}$$

According to the amplitude–frequency formulation, the above residuals can be rewritten in the forms of weighted residuals:

$$R_{11} = \frac{4}{T_1}\int_0^{\frac{T_1}{4}} R_1 \cos(t)dt\,,\ T_1 = 2\pi \tag{23}$$

And:

$$R_{22} = \frac{4}{T_2}\int_0^{\frac{T_2}{4}} R_2 \cos(\omega t)dt\,,\ T_2 = \frac{2\pi}{\omega} \tag{24}$$

Applying He's frequency-amplitude formulation:

$$\omega^2 = \frac{\omega_1{}^2 R_{22} - \omega_2{}^2 R_{11}}{R_{22} - R_{11}} \tag{25}$$

Where:

$$\omega_1 = 1,\ \omega_2 = \omega \tag{26}$$

Consider the nonlinear oscillator Eq. (1). Its weighted residuals, therefore, can be written in the form:

$$R_{11} = \frac{2}{\pi}\left(\frac{3}{16}A^3 a\pi + \frac{5}{32}A^5 b\pi + \frac{35}{256}A^7 c\pi\right) \tag{27}$$

And:

$$R_{22} = \frac{A}{\pi}\left(\frac{1}{2}\pi - \frac{1}{2}\omega^2\pi + \frac{35}{128}cA^6\pi + \frac{3}{8}aA^2\pi + \frac{5}{16}bA^4\pi\right) \tag{28}$$

The approximate frequency can be obtained:

$$\omega_{AFF} = \sqrt{1 + \frac{3}{4}aA^2 + \frac{5}{8}bA^4 + \frac{35}{64}cA^6} \tag{29}$$

For the case $a = 1,\ b = c = 0$, Eq. (1) turns to be the well-known Duffing equation which agrees exactly with our prediction. Similarly for the Amplitude Frequency Formulation, the relative error is:

$$\lim_{A\to\infty} \frac{\omega_{AFF}}{\omega_{Exact}} = 1.0956 \tag{30}$$

The lowest order approximation given by (29) is actually within 9.5% of the exact frequency.

The application of the Variational Approach (VA)

The Variational Approach for nonlinear oscillators was proposed (He, 2007): Consider the nonlinear oscillator Eq. (3). Its variational principle can be obtained by using the semi inverse method:

$$J(u) = \int_0^{\frac{T}{4}} \left\{-\frac{1}{2}u'^2 + F(u_2, u_2', u_2'')\right\} dt \tag{31}$$

Where T is period of the oscillator $f = \dfrac{\partial F}{\partial u}$.

Assume that its approximate solution can be expressed as:

$$u(t) = A\cos(\omega t) \tag{32}$$

Substituting Eq. (3) into Eq. (2) and applying the Ritz-like method (He, 2007), (Kaya et al, 2010), the approximate frequency can be obtained. In Ref. (Liu, 2009), Liu obtained the approximate frequency for Eq. (3) by Variational Approach:

$$\omega_{VA} = \sqrt{1+\frac{3}{4}aA^2+\frac{5}{8}bA^4+\frac{35}{64}cA^6} \tag{33}$$

This result is the same as the solutions by the Amplitude-Frequency Formulation, Integral Iteration Method And Hamiltonian Approach, which were shown to have high accuracy.

Results and discussions

In order to demonstrate the efficiency of the present methods, comparison of approximate results with each other, is shown in Table (1), Table (2) and Table (3) also the results are compared with an accurate numerical solution, using fourth-order Runge-Kutta. A very interesting agreement between the results is observed, which confirms the excellent validity of the proposed methods.

Table 1. Comparison between approximate frequencies those are obtained by presented methods and with fourth-order Runge-Kutta. $(a=1, b=1, c=0)$

A	ω_{CHV}	ω_{AFF}	ω_{IIM}	ω_{VA}	ω_{R-K}
0.01	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	1.1069	1.1075	1.1075	1.1075	1.1061
1.00	1.5343	1.5411	1.5411	1.5411	1.5257
2.00	3.6968	3.7417	3.7417	3.7417	3.6401
3.00	7.5291	7.6404	7.6404	7.6404	7.3905
5.00	19.9337	20.2577	20.2577	20.2577	19.5885
10.0	78.2155	79.5362	79.5362	79.5362	76.6201

Table 2. Comparison between approximate frequencies those are obtained by presented methods and with fourth-order Runge-Kutta. $(a=1, b=0, c=1)$

A	ω_{CHV}	ω_{AFF}	ω_{IIM}	ω_{VA}	ω_{R-K}
0.01	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	1.1069	1.1075	1.1075	1.1075	1.1061
1.00	1.5343	1.5411	1.5411	1.5411	1.5257
2.00	3.6968	3.7417	3.7417	3.7417	3.6401
3.00	7.5291	7.6404	7.6404	7.6404	7.3905
5.00	19.9337	20.2577	20.2577	20.2577	19.5885
10.0	78.2155	79.5362	79.5362	79.5362	76.6201

Table 3. Comparison between approximate frequencies those are obtained by presented methods and with fourth-order Runge-Kutta. $(a=1, b=1, c=1)$

A	ω_{CHV}	ω_{AFF}	ω_{IIM}	ω_{VA}	ω_{R-K}
0.01	1.0000	1.0000	1.0000	1.0000	1.0000
0.50	1.1105	1.1114	1.1114	1.1114	1.1092
1.00	1.6925	1.7093	1.7093	1.7093	1.6597
2.00	6.8069	7.0000	7.0000	7.0000	6.5203
3.00	20.7070	21.3787	21.3787	21.3787	19.0125
5.00	91.5020	94.6324	94.6324	94.6324	86.6528
10.0	714.4877	743.7748	743.7748	743.7748	687.0134

The numerical results obtained by presented methods are illustrated in Tables 1-3. Numerical results validate accuracy of the solution techniques also it is shown in Tables 1-3, validity of all the solution techniques is guaranteed for stronger nonlinearities. It is found that, the relative error is smaller for Coupled Homotopy Variational Formulation.

Conclusions

In this paper, different methods for solving the problem of nonlinear oscillators have been analyzed. After comparing approximate results with the numerical ones by means of tabulated values in all cases, excellent agreement proved that these methods is powerful and reliable to solving nonlinear oscillatory systems with high nonlinearity. The second or higher order approximates of course can readily obtained with high accuracy but the solution procedure for the second or higher order approximate solution require more computational procedure.

The obtained results of Amplitude Frequency Formulation, Integral Iteration Method and Varitional Approach for the first approximate are similar also it is found that the relative error is smaller for Coupled Homotopy Variational Formulation. As presented in Refs. (He, 2010), without any cumbersome procedure, the second or higher order approximates of course can readily obtained with high accuracy. Comparison of the result which is obtained by these methods with the obtained result by the exact solution reveals that these approaches are very effective and convenient.

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