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Cohen-Macaulay of ideal $I_2(G)$

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Abstract
In this paper, we study the Cohen-Macaulay of ideal $I_2(G)$, where $I_2(G)=\langle xyz \mid x\text{-}y\text{-}z \text{ is 2-path in } G\rangle$. Also, we determined the 2-projective dimension R-module, $R/I_2(G)$ denoted by $pd_2(G)$, of some graphs.

Keywords: Cohen-Macaulay-projective dimension-ideal-path.

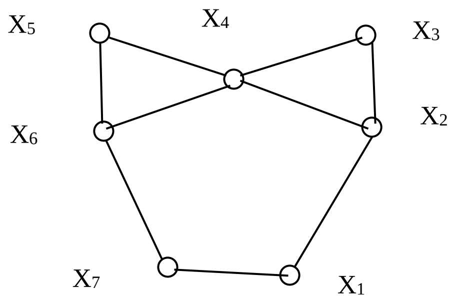
Introduction

A simple graph is a pair G=(V,E), where V=V(G) and E=E(G) are the sets of vertices and edges of G, respectively. A walk is an alternating sequence of vertices and connecting edges. A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A path with length n denotes by P_n. In a graph G, the distance between two distinct vertices x and y, denoted by d(x,y), is the length of the shortest path connecting x and y, if such a path exists: otherwise, we set d(x,y)=∞. The diameter of a graph G is diam(G)=sup{ d(x,y) : x and y are distinct vertices of G}.. Also, a cycle is a path that begins and ends on the same vertex. A cycle with length n denotes by C_n. A graph G is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. For a positive integer r, a complete r-partite graph is one in which each vertex is joined to every vertex that is not in the some subset. The complete bipartite graph with part sizes m and n is denoted by $K_{m,n}$. The graph $K_{۱,n-۱}$ is called a star graph in which the vertex with degree n-1 is called the center of the graph. For any graph G, we denote N[x]={ y∈ V(G) : (x,y) is an edge of G}.Recall that the projective dimension of an R-module M, denoted by pd(M) , is the length of the minimal free resolution of M, that is, pd(M)=max{I | $\beta_{i,j}$ (M)≠0 for some j}. There is a strong connection between the topology of the simplicial complex and the structure of the free resolution of K[Δ]. Let $\beta_{i,j}(\Delta)$ $ denotes the N-graded Betti numbers of the Stanley-Reisner ring K[Δ].
To any finite simple graph G with the vertex set V(G)={$x_1,\dots,x_n$} and the edge set E(G), one can attach an ideal in the Polynomial rings R=K [$x_1,\dots,x_n$] over the field K, where ideal I_2(G) is called the edge ideal of G such that I_2(G)=<xyz | x-y-z is 2-path in G>. Also the edge ring of G, denoted by K(G) is defined to be the quotient ring K(G)=R/I_2(G). Edge ideals and edge rings were first introduced by Villarreal [5] and then they have been studied by many authers in order to examine their algebraic properties according to the combinatorial data of graphs. In this paper, we denote S_n for a star graph with n+1 vertices.

Cohen-Macaulay of ideal I_2(G) and pd_2(G) of some graph G

Definition 1. Let G be a graph with vertex set V. Then a subset $A \subseteq V$ is a 2-vertex cover for G if for every path xyz of G we have $\{x,y,z\} \cap A \neq \emptyset$. A 2-minimal vertex cover of G is a subset A of V such that A is a 2-vertex cover, and no proper subset of A is a vertex cover for G. The smallest cardinality of a 2-vertex cover of G is called the 2-vertex covering number of G and is denoted by $\alpha_{02}(G)$.

Example 2. Let G be a graph shown in the figure. Then the set $\{x_2, x_4, x_7\}$ is a 2-minimal vertex cover of G and $\alpha_{02}(G) = 3$.



Definition ۳. Let G be a graph with vertex set V. A subset $A \subseteq V$ is a k-independent if for even x of S we have $deg_{G[S]} x \leq k-1$. The maximum possible cardinality of an k-independent set of G, denoted $\beta_{0k}(G)$, is called the k-independence number of G. It is easy see that

$$\alpha_{02}(G) + \beta_{0k}(G) = |V(G)|.$$

Definition 4. Let G be a graph without isolated vertices, Let $S = K[x_1, \cdots, x_n]$ the polynomial ring on the vertices of G over some fixed field K. The 2-pathes ideal $I_2(G)$ associated to the graph G is the ideal of S generated by the set of squar-free monomials $x_i x_j x_r$ such that $v_i v_j v_r$ is the path of G, that is $I_2(G) =< \{x_i x_j x_r \mid \{v_i v_j v_r\} \in P_3(G)\} > \subseteq S$.

Proposition 5. Let $S = K[x_1, \cdots, x_n]$ be a polynomial ring over a field K and G a graph with vertices $v_1, \cdots, v_n$. If P is an ideal of R generated by $A = \{x_{i_1}, \cdots, x_{i_k}\}$, then P is a minimal prime of $I_2(G)$ if and only if A is a 2-minimal vertex cover of G.
Proof. It is easy see that $I_2(G) \subseteq P$ if and only if A is a 2-vertex cover of G. Now, let A is a 2-minimal vertex cover of G. By Proposition 5.1.3 [5] any minimal prime ideal of $I_2(G)$ is a face ideal thus P is a minimal prime of $I_2(G)$. The convers is clear.

Corollary 6. If G is a graph and $I_2(G)$ its 2-path ideal, then

$$ht(I_2(G)) = \alpha_{02}(G).$$

Proof. If follows from Proposition 5 and the definition of $\alpha_{02}(G)$.

Definition 7. A graph G is 2-unmixed if all of its 2-minimal vertex covers have the same cardinality.

Definition 8. A graph G with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ is 2-cohen-Macaullay over field K if the quotient ring $K[x_1,\ldots,x_n]/I_2(G)$ is cohen-Macaulay.

Proposition 9. If G is a 2-cohen-Macaulay graph, then G is 2-unmixed.
Proof. By corollary 1.3.6 [5], $I_2(G) = \bigcap_{P \in \min(I_2(G))} P$. Since $R/I_2(G)$ is cohen-Macaullay, all minimal prime ideals of $I_2(G)$ have the same height. Then by Proposition 5, all 2-minimal vertex cover of G have the same cardinality, as desired.

Proposition 10. If G is a graph and $G_1, \cdots, G_s$ its connected components, then G is 2-cohen-Macaulay if and only if for all i, G_i is cohen-Macaulay.
Proof. Let $R = K[V(G)]$ and $R_i = L[V(G_i)]$ for all i. Since

$$R/I_2(G) \cong R_1/I_2(G_1) \otimes_K \cdots \otimes_K R_s/I_2(G_s),$$

Hence the results follow from Corollary 2.2.22 [5].

Definition 11. For any graph G one associates the complementary scimplical complex $\Delta_2(G)$, which is defined as

$$\Delta_2(G) := \{A \subset V \mid A \text{ is } 2-independent\ set\ in\ G\}.$$

This means that the facets of $\Delta_2(G)$ are precisely the maximal 2-independent sets in G, that is the complements in V of the minimal 2-vertex covers. Thus $\Delta_2(G)$ precisely the Stanley-Reisnercomplex of $I_2(G)$.
It is easy see that $\omega(\Delta_2(G)) = \{\{x,y,z\} \mid xyz \in P_3(G)\}$. Therefore $I_2(G) = I_{\Delta_2(G)}$, and so G is 2-C-M graph if and only if the simplicial complex $\Delta_2(G)$ is cohen-Macaulay.

Now, we can show the following proposition.

Proposition 12. The following statements hold

a) For any $n \geq 1$ the complete graph K_n is cohen-Macaulay.

b) The complete bipartite graph $K_{m,n}$ is cohen-Macaulay if and only if $m + n \leq 4$.

Proof. a) Since $\Delta_2(K_n) = \lessdot \{x,y\} \mid x,y \in V(K_n) \gtrdot$, thus $\Delta_2(K_n)$ is connected 1-dimensional simplicial complex, then by Corollary 5.3.7 [5],$\Delta_2(K_n)$ is cohen-Macaulay so K_n is cohen-Macaulay.
b)If $m + n \leq 4$, then $K_{m,n} \cong P_2, P_4, C_4$, then it is easy to see that $\Delta_2(K_{m,n})$ is c so ohen-Macaulay $K_{m,n}$ is cohen-Macaulay.
Conversely, let $K_{m,n}$ is cohen-Macaulay and $m + n \geq 5$. Take $V_1 = \{x_1, \cdots, x_n\}$ and $V_2 = \{y_1, \cdots, y_m\}$ are the partie sets of $K_{m,n}$. One has

$\Delta_2(K_{m,n}) =< \{x_1, \cdots, x_n\}, \{y_1, \cdots, y_m\}, \{x_i, y_j\} \mid 1 \leq i \leq n, 1 \leq j \leq m >.$

Since $m + n \geq 5$, hence $\Delta_2(K_{m,n})$ is not pure simplicial complex, then by 5.3.12 [5]$\Delta_2(K_{m,n})$ is not cohen-Macaulay. Which is a contradiction, as desired.

Now, we present a result about the Hilbert series of $K[\Delta_2(K_n)]$ and $K[\Delta_2(K_{m,n})]$.

Proposition 13. If $\Delta_2(K_n)$ and $\Delta_2(K_{m,n})$ are the complementary simplicial complexes K_n and $K_{m,n}$ respectively, then

c) $F(K[\Delta_2(K_n), z) = 1 + nz/(1-z) + n(n-1)/2(1-z)^2$

d) $F(K[\Delta_2(K_n), z) = 1/(1-z)^n + 1/(1-z)^m + m.n\, z^2/(1-z)^2 - 1.$

Proof. a) Since $\Delta_2(K_n) = \lessdot \{x,y\} \mid x,y \in V(K_n) \gtrdot$ hence $dim\, \Delta_2(K_n) = 1$ and $f_{-1}(K_n) = 1$, $f_0(K_n) = n$ and $f_1(K_n) = \binom{n}{2} = n(n-1)/2$. By Corollary 5.4.5 [5]. We have

$F(K[\Delta_2(K_n), z) = 1 + nz/1 - z + n(n-1)/2 . z^2/2(1-z)^2$

b)Let $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_m\}$ are the parties sets of $K_{m,n}$. Since

$$\Delta_2(K_{m,n}) =< \{x_1, \cdots, x_n\}, \{y_1, \cdots, y_m\}, \{x_i, y_j\} \mid 1 \leq i \leq n, 1 \leq j \leq m >.$$

Then it is easy see that $f_1\left(\Delta_2(K_{m,n})\right) = f_1\left(\Delta(K_{m,n})\right) + mn$ and $f_i\left(\Delta_2(K_{m,n})\right) = f_i\left(\Delta(K_{m,n})\right)$ for all $i \neq 1$. In the other hand by 6.6.6 [5], $F(K[\Delta_2(K_n), z) = 1/(1-z)^n + 1/(1-z)^m - 1$. Thus

$$F(K[\Delta_2(K_n), z) = 1/(1-z)^n + 1/(1-z)^m + m.n\, z^2/(1-z)^2 - 1$$

As desired.

Corollary 14. $F(K[\Delta_2(S_n), z) = 1/(1-z)^n + nz^2/(1-z)^2 + z/(1-z)$.

Proof. It follows from Proposition 13 with assume $m = 1$.

In this section we mainly present basic properties of 2-shellable graphs.

Lemma 15. Let G be a graph and x be a vertex of degree 1 in G and let $y \in N(x)$ and $G' = G - (\{y\} \cup N(y))$. Then $\Delta_2(G') = lK_{\Delta_2(G)}(\{x,y\})$. Moreover F is a facet of $\Delta_2(G')$ if and only if $F \cup \{x,y\}$ is a facet of $\Delta_2(G)$.
Proof. a) Let $F \in lK_{\Delta_2(G)}(\{x,y\})$. Then $F \in \Delta_2(G)$, $x, y \notin F$ and $F \cup \{x,y\} \in \Delta_2(G)$. This implies that $(F \cup \{x,y\}) \cap N[y] = \emptyset$ and so $F \subseteq (V - \{x,y\}) \cup N[y] = (V - \{y\}) \cup N[y] = V(G')$. Thus F is 2-independent in G', it follows that $F \in \Delta_2(G')$. Conversely let $F \in \Delta_2(G')$, then F is 2-independent in G' and $F \cap (\{x\} \cup [y]) = \emptyset$. Therefore $F \cup \{x,y\}$ is 2-independent in G and so $F \cup \{x,y\} \in \Delta_2(G), F \cup \{x,y\} = \emptyset$. Thus $F \in lK_{\Delta_2(G)}(\{x,y\})$. Finally from part one follows that F is a Facet of $\Delta_2(G')$ if and only if $F \cup \{x,y\}$ is a facet of $\Delta_2(G)$.

Definition 16. Fix a field K, and set $R = K[x_1, \cdots, x_n]$. If G is a graph with vertex set $V(G) = \{x_1, x_2, \cdots, x_n\}$, we define the projective dimension of G to be the 2-projective dimension R-modul $R/I_2(G)$, and we will write $pd_2(G) = pd(R/I_2(G))$.

Proposition 17. If G is a graph and $\{x,y\}$ is a edge of G, then

$$P_2(G) \leq max\{P_2(G - (N[x] \cup N[y])) + \deg(x) + \deg(y) - |N[x] \cap N[y]|, P_2(G - x) + 1, P_2(G - y) + 1\}$$

Proof. Let $N[x] = \{x_1, \cdots, x_s\}$ and $N[y] = \{y_1, \cdots, y_r\}$. It is easy to see that

$$I_2(G): xy = (I_2(G) - (N[x] \cup N[y]), x_1, \cdots, x_s, y_1, \cdots, y_r).$$

Now, let

$$R' = K[V(G - (N[x] \cup N[y])],$$

then

$$depth(R/I_2(G): xy) = depth(R'/I_2(G - (N[x] \cup N[y])$$

And so by Auslander-Buchsbaum formula, we have

$$pd_2(R/I_2(x): xy) = pd_2(G - (N[x] \cup N[y]) + \deg(x) + \deg(y) - |N[x] \cap N[y]|,$$

$$pd_2(R/(I_2(G), x)) = pd_2(G - x) + 1 \qquad and \qquad pd_2(R/(I_2(x), y) = pd_2(G - y) + 1.$$

On the other hand by Proposition 10, together with the exact sequence

$$0 \to R/I_2(G): xy \to R/I_2(G) \to R/I_2(G)xy \to 0,$$

follows that,

$$P_2(G) \leq max\{P_2(G - (N[x] \cup N[y])) + \deg(x) + \deg(y) - |N[x] \cap N[y]|, P_2(G - x) + 1, P_2(G - y) + 1\}$$

Proposition 18. Let G be a graph and $I_2(G)$ is path ideal of G. Then

$$Bight(I_2(G)) \leq pd_2(G).$$

Proof. Let P be a minimal vertex cover with maximal cardinality. Then by Proposition 5, P is an associated prime of $R/I_2(G)$, so

$$pd_2(G) = pd(R/I_2(G)) \geq pd_{R_p}(R_p/I_2(G)R_p) = dimR_p = htP.$$

Proposition 19. Let K_n denote the complete graph on n vertices and let $K_{m,n}$ denote the complete bipartite graph on $m+n$ vertices.

a) $pd_2(K_n) = n-2$
b) $pd_2(K_{m,n}) = m+n-2$.

Proof. a) The proof is by induction on n. If $n = 2\ or\ 3$, the result easy follows. Let $n \geq 4$ and suppose that for every complete graphs K_n of other less than n the result is true. Since $Bight(I_2(K_n)) = n-2$ then by Proposition $pd_2(K_n) \geq n-2$. On the other hand by the inductive hypothesis, we have $pd_2(K_{n-1}) = n-3$, so by Proposition 17

$$pd_2(K_n) \leq max\{n-2, n-2\},$$

this completes the proof.

b) Again we use by induction on $m+n$. If $n+m = 2\ or\ 3$, then it is easy to see that $pd_2(K_{m,n}) = 0\ or\ 1$. Let $n+m \geq 4$ and suppose that for every complete bipartite graph $K_{m,n}$ of order less than $m+n$ the result is true. Since $Bight\left(I_2(K_{m,n})\right) = m+n-2$ then $pd_2(K_{m,n}) \geq m+n-2$. Also, by the inductive hypothesis we have $pd_2(K_{m-1,n}) = m+n-3$ and $pd_2(K_{m,n-1}) = m+n-3$. So by Proposition 17,

$$pd_2(K_{m,n}) \leq max\{m+n-2, pd_2(K_{m-1,n})+1, pd_2(K_{m,n-1})+1\} = m+n-2.$$

As desired.

Corollary 20. Let S_n denote the star graph on $n+1$ vertices and $S_{m,n}$ denote the double star, then $pd_2(S_{m,n}) = m+n$.

Proof. It follows from Proposition 19, with assume $m = 1$ and it is easy to see that $Bight\ I_2(S_{m,n}) = m+n$ and so by Proposition 17, it follows that $pd_2(S_{m,n}) = m+n$.

Sample References

H. Dao, C. Huneke, and J. Schweig, Projective dimension and independence complex homology. In preparation.
A.Dochtermann and A. Engstrom, Algebraic properties of edge via combinatorial topology. Electron. J. Combin. 16 (2009), Special volume in honor of Anders Bjorner, Reseearch Paper 2.
C. Peskine and L. Szpiro, Dimension projective finite et cohomologie locale, Inst. Hautes Etudes Sci. Publ. Math. **42** (1973) 47-119.
N. Terai, Alexander duality theorem and Stanley-Reisner rings. Free resolutions of coordinate rings of projective varieties and related topics (Japanese) (Kyoto, 1998). Surikaisekikenkyusho Kokyurokn No. 1078 (1999), 174-184.
R. Villarreal, Monomial Algebras, Ssecond edition, March 26, 2015 by chapman and Hall/CRC.