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### **Cohen-Macaulay of ideal I**<sub>2</sub>(G)

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#### Abstract

In this paper, we study the Cohen-Macaulay of ideal I<sub>2</sub>(G), where  $I_2(G) = \langle xyz | x-y-z \text{ is } 2\text{-path in } G \rangle$ . Also, we determined the 2projective dimension R-module, R/I2(G) denoted by pd2(G), of some graphs.

Keywords: Cohen-Macaulay-projective dimension-ideal-path.

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Introduction

A simple graph is a pair G=(V,E), where V=V(G) and E=E(G) are the sets of vertices and edges of G, respectively. A walk is an alternating sequence of vertices and connecting edges. A path is a walk that does not include any vertex twice, except that its first vertex might be the same as its last. A path with length n denotes by P\_n. In a graph G, the distance between two distinct vertices x and y, denoted by d(x,y), is the length of the shortest path connecting x and y, if such a path exists: otherwise, we set  $d(x,y)=\infty$ . The diameter of a graph G is diam(G)=sup{ d(x,y) : x and y are distinct vertices of G}. Also, a cycle is a path that begins and ends on the same vertex. A cycle with length n denotes by C<sub>n</sub>. A graph G is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use K<sub>n</sub> to denote the complete graph with n vertices. For a positive integer r, a complete r-partite graph is one in which each vertex is joined to every vertex that is not in the some subset. The complete bipartite graph with part sizes m and n is denoted by  $K_{m,n}$ . The graph  $K_{1,n-1}$  is called a star graph in which the vertex with degree n-1 is called the center of the graph. For any graph G, we denote  $N[x] = \{y \in V(G) : (x,y) \text{ is an edge} \}$ of G}.Recall that the projective dimension of an R-module M, denoted by pd(M), is the length of the minimal free resolution of M, that is,  $pd(M)=max\{I \mid \beta_{i,j}(M)\neq 0 \text{ for some } j\}$ . There is a strong connection between the topology of the simplicial complex and the structure of the free resolution of K[ $\Delta$ ]. Let  $\beta_{i,i}(\Delta)$  \$ denotes the N-graded Betti numbers of the Stanley-Reisner ring  $K[\Delta]$ .

To any finite simple graph G with the vertex set  $V(G)=\{x_1,...,x_n\}$  and the edge set E(G), one can attach an ideal in the Polynomial rings  $R=K[x_1,...,x_n]$  over the field K, where ideal  $I_2(G)$  is called the edge ideal of G such that  $I_2(G)=\langle xyz | x-y-z \text{ is } 2\text{-path in } G \rangle$ . Also the edge ring of G, denoted by K(G) is defined to be the quotient ring  $K(G)=R/I_2(G)$ . Edge ideals and edge rings were first introduced by Villarreal [5] and then they have been studied by many authers in order to examine their algebraic properties according to the combinatorial data of graphs. In this paper, we denote  $S_n$  for a star graph with n+1 vertices.

#### Cohen-Macaulay of ideal I<sub>2</sub>(G) and pd<sub>2</sub>(G) of some graph G

**Definition 1.** Let *G* be a graph with vertex set *V*. Then a subset  $A \subseteq V$  is a 2-vertex cover for *G* if for every path xyz of *G* we have  $\{x, y, z\} \cap A \neq \emptyset$ . A 2-minimal vertex cover of *G* is a subset *A* of *V* such that *A* is a 2-vertex cover, and no proper subset of *A* is a vertex cover for *G*. The smallest cardinality of a 2-vertex cover of *G* is called the 2-vertex covering number of *G* and is denoted by  $\alpha_{02}(G)$ .

**Example 2.** Let G be a graph shown in the figure. Then the set  $\{x_2, x_4, x_7\}$  is a 2-minimal vertex cover of G and  $\alpha_{02}(G) = 3$ .



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**Definition**  $\mathcal{V}$ . Let G be a graph with vertex set V. A subset  $A \subseteq V$  is a k-independent if for even x of S we have  $deg_{G[S]}x \leq k-1$ . The maximum possible cardinality of an k-independent set of G, denoted  $\beta_{0k}(G)$ , is called the k-independence number of G. It is easy see that

$$\alpha_{02}(G) + \beta_{0k}(G) = |V(G)|.$$

**Definition 4.** Let *G* be a graph without isolated vertices, Let  $S = K[x_1, \dots, x_n]$  the polynomial ring on the vertices of *G* over some fixed field *K*. The 2-pathes ideal  $I_2(G)$  associated to the graph *G* is the ideal of *S* generated by the set of squar-free monomials  $x_i x_j x_r$  such that  $v_i v_j v_r$  is the path of *G*, that is  $I_2(G) = \langle \{x_i x_j x_r \mid \{v_i v_j v_r\} \in P_3(G)\} \rangle \subseteq S$ .

**Proposition 5.** Let  $S = K[x_1, \dots, x_n]$  be a polynomial ring over a field K and G a graph with vertices  $v_1, \dots, v_n$ . If P is an ideal of R generated by  $A = \{x_{i_1}, \dots, x_{i_k}\}$ , then P is a minimal prime of  $I_2(G)$  if and only if A is a 2-minimal vertex cover of G.

Proof. It is easy see that  $l_2(G) \subseteq P$  if and only if A is a 2-vertex cover of G. Now, let A is a 2-minimal vertex cover of G. By Proposition 5.1.3 [5] any minimal prime ideal of  $l_2(G)$  is a face ideal thus P is a minimal prime of  $l_2(G)$ . The convers is clear.

**Corollary 6.** If G is a graph and  $I_2(G)$  its 2-path ideal, then

 $ht(I_2(G)) = \alpha_{02}(G).$ 

Proof. If follows from Proposition 5 and the definition of  $\alpha_{02}(G)$ .

**Definition 7.** A graph *G* is 2-unmixed if all of its 2-minimal vertex covers have the same cardinality.

**Definition 8.** A graph *G* with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$  is 2-cohen-Macaullay over field *K* if the quotient ring  $K[x_1, \dots, x_n]/I_2(G)$  is cohen-Macaulay.

**Proposition 9.** If *G* is a 2-cohen-Macaulay graph, then *G* is 2-unmixed. Proof. By corollary 1.3.6 [5],  $I_2(G) = \bigcap_{P \in \min(I_2(G))} P$ . Since  $R/I_2(G)$  is cohen-Macaullay, all minimal prime ideals of  $I_2(G)$  have the same height. Then by Proposition 5, all 2-minimal vertex cover of *G* have the same cardinality, as desired.

**Proposition 10.** If G is a graph and  $G_1, \dots, G_s$  its connected components, then G is 2-cohen-Macaulay if and only if for all *i*,  $G_i$  is cohen-Macaulay. Proof. Let R = K[V(G)] and  $R_i = L[V(G_i)]$  for all *i*. Since

$$R/I_2(G) \cong R_1/I_2(G_1) \bigotimes_K \cdots \bigotimes_K R_s/I_2(G_s),$$
  
Hence the results follow from Corollary 2.2.22 [5].

**Definition 11.** For any graph *G* one associates the complementary scimplical complex  $\Delta_2(G)$ , which is defined as  $\Delta_2(G) := \{A \subset V \mid A \text{ is } 2 - independent set in G\}.$ 

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This means that the facets of  $\Delta_2(G)$  are precisely the maximal 2-independent sets in G, that is the complements in V of the minimal 2-vertex covers. Thus  $\Delta_2(G)$  precisely the Stanley-Reisnercomplex of  $I_2(G)$ .

It is easy see that  $\omega(\Delta_2(G)) = \{\{x, y, z\} \mid xyz \in P_3(G)\}$ . Therefore  $I_2(G) = I_{\Delta_2(G)}$ , and so G is 2-C-M graph if and only if the simplicial complex  $\Delta_2(G)$  is cohen-Macaulay.

Now, we can show the following proposition.

Proposition 12. The following statements hold

- a) For any  $n \ge 1$  the complete graph  $K_n$  is cohen-Macaulay.
- b) The complete bipartite graph  $K_{m,n}$  is cohen-Macaulay if and only if  $m + n \le 4$ .

Proof. a) Since  $\Delta_2(K_n) = \langle \{x, y\} | x, y \in V(K_n) \rangle$ , thus  $\Delta_2(K_n)$  is connected 1-dimensional simplicial complex, then by Corollary 5.3.7 [5],  $\Delta_2(K_n)$  is cohen-Macaulay so  $K_n$  is cohen-Macaulay. b) If  $m + n \leq 4$ , then  $K_{m,n} \cong P_2, P_4, C_4$ , then it is easy to see that  $\Delta_2(K_{m,n})$  is c so ohen-Macaulay  $K_{m,n}$  is cohen-Macaulay.

Conversely, let  $K_{m,n}$  is cohen-Macaulay and  $m + n \ge 5$ . Take  $V_1 = \{x_1, \dots, x_n\}$  and  $V_2 = \{y_1, \dots, y_m\}$  are the partie sets of  $K_{m,n}$ . One has

$$\Delta_2(K_{m,n}) = <\{x_1, \cdots, x_n\}, \{y_1, \cdots, y_m\}, \{x_i, y_j\} \mid 1 \le i \le n, 1 \le j \le m >.$$

Since  $m + n \ge 5$ , hence  $\Delta_2(K_{m,n})$  is not pure simplicial complex, then by 5.3.12 [5] $\Delta_2(K_{m,n})$  is not cohen-Macaulay. Which is a contradiction, as desired.

Now, we present a result about the Hilbert series of  $K[\Delta_2(K_n)]$  and  $K[\Delta_2(K_{m,n})]$ .

**Proposition 13.** If  $\Delta_2(K_n)$  and  $\Delta_2(K_{m,n})$  are the complementary simplicial complexes  $K_n$  and  $K_{m,n}$  respectively, then

c) 
$$F(K[\Delta_2(K_n), z) = 1 + nz/(1-z) + n(n-1)/2(1-z)^2$$

d) 
$$F(K[\Delta_2(K_n), z) = 1/(1-z)^n + 1/(1-z)^m + m \cdot n \cdot z^2/(1-z)^2 - 1.$$

Proof. a) Since  $\Delta_2(K_n) = \langle \{x, y\} | x, y \in V(K_n) \rangle$  hence  $\dim \Delta_2(K_n) = 1$  and  $f_{-1}(K_n) = 1, f_0(K_n) = n$  and  $f_1(K_n) = \binom{n}{2} = n(n-1)/2$ . By Corollary 5.4.5 [5]. We have

$$F(K[\Delta_2(K_n), z) = 1 + nz/1 - z + n(n-1)/2 \cdot z^2/2(1-z)^2$$

b)Let  $\{x_1, \dots, x_n\}$  and  $\{y_1, \dots, y_m\}$  are the parties sets of  $K_{m,n}$ . Since

$$\Delta_2 \Big( K_{m,n} \Big) = < \{ x_1, \cdots, x_n \}, \{ y_1, \cdots, y_m \}, \big\{ x_i, y_j \big\} \mid 1 \le i \le n \,, \, 1 \le j \le m >.$$

Then it is easy see that  $f_1(\Delta_2(K_{m,n})) = f_1(\Delta(K_{m,n})) + mn$  and  $f_i(\Delta_2(K_{m,n})) = f_i(\Delta(K_{m,n}))$  for all  $i \neq 1$ . In the other hand by 6.6.6 [5],  $F(K[\Delta_2(K_n), z) = 1/(1-z)^n + 1/(1-z)^m - 1$ . Thus

$$F(K[\Delta_2(K_n), z) = 1/(1-z)^n + 1/(1-z)^m + m \cdot n \cdot z^2/(1-z)^2 - 1$$

As desired.

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Corollary 14.  $F(K[\Delta_2(S_n), z) = 1/(1-z)^n + nz^2/(1-z)^2 + z/(1-z).$ 

Proof. It follows from Proposition 13 with assume m = 1.

In this section we mainly present basic properties of 2-shellable graphs.

Lemma 15. Let *G* be a graph and *x* be a vertex of degree 1 in *G* and let  $y \in N(x)$  and  $G' = G - (\{y\} \cup N(y))$ . Then  $\Delta_2(G') = lK_{\Delta_2(G)}(\{x, y\})$ . Moreover F is a facet of  $\Delta_2(G')$  if and only if  $F \cup \{x, y\}$  is a facet of  $\Delta_2(G)$ . Proof. a) Let  $F \in lK_{\Delta_2(G)}(\{x, y\})$ . Then  $F \in \Delta_2(G)$ ,  $x, y \notin F$  and  $F \cup \{x, y\} \in \Delta_2(G)$ . This implies that  $(F \cup \{x, y\}) \cap N[y] = \emptyset$  and so  $F \subseteq (V - \{x, y\}) \cup N[y] = (V - \{y\}) \cup N[y] = V(G')$ . Thus *F* is 2-independent in *G'*, it follows that  $F \in \Delta_2(G')$ . Conversely let  $F \in \Delta_2(G')$ , then *F* is 2independent in *G'* and  $F \cap (\{x\} \cup [y]) = \emptyset$ . Therefore  $F \cup \{x, y\}$  is 2-independent in *G* and so  $F \cup \{x, y\} \in \Delta_2(G), F \cup \{x, y\} = \emptyset$ . Thus  $F \in lK_{\Delta_2(G)}(\{x, y\})$ . Finally from part one follows that *F* is a Facet of  $\Delta_2(G')$  if and only if  $F \cup \{x, y\}$  is a facet of  $\Delta_2(G)$ .

**Definition 16.** Fix a field *K*, and set  $R = K[x_1, \dots, x_n]$ . If *G* is a graph with vertex set  $V(G) = \{x_1, x_2, \dots, x_n\}$ , we define the projective dimension of *G* to be the 2-projective dimension *R*-modul  $R/I_2(G)$ , and we will write  $pd_2(G) = pd(R/I_2(G))$ .

**Proposition 17.** If *G* is a graph and  $\{x, y\}$  is a edge of *G*, then

$$\begin{split} P_2(G) &\leq max \big\{ P_2\big(G - (N[x] \cup N[y])\big) + \deg(x) + \deg(y) - |N[x] \cap N[y]|, P_2(G - x) \\ &+ 1, P_2(G - y) + 1 \big\} \end{split}$$

Proof. Let  $N[x] = \{x_1, \dots, x_s\}$  and  $N[y] = \{y_1, \dots, y_r\}$ . It is easy to see that

$$I_2(G): xy = (I_2(G) - (N[x] \cup N[y]), x_1, \dots, x_s, y_1, \dots, y_r)$$

Now, let

$$R' = K[V(G - (N[x] \cup N[y])],$$

then

$$depth(R/I_2(G):xy) = depth(R'/I_2(G - (N[x] \cup N[y]))$$

And so by Auslander-Buchsbaum formula, we have

$$pd_2(R/I_2(x):xy) = pd_2(G - (N[x] \cup N[y]) + \deg(x) + \deg(y) - |N[x] \cap N[y]|,$$

$$pd_2(R/(I_2(G), x)) = pd_2(G - x) + 1$$
 and  $pd_2(R/(I_2(x), y)) = pd_2(G - y) + 1$ 

On the other hand by Proposition 10, together with the exact sequence

$$0 \rightarrow R/I_2(G) {:} xy \rightarrow R/I_2(G) \rightarrow R/I_2(G) xy \rightarrow 0,$$

follows that,

$$\begin{aligned} P_2(G) &\leq max \Big\{ P_2 \Big( G - (N[x] \cup N[y]) \Big) + \deg(x) + \deg(y) - |N[x] \cap N[y]|, P_2(G - x) \\ &+ 1, P_2(G - y) + 1 \Big\} \end{aligned}$$

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**Proposition 18.** Let G be a graph and  $I_2(G)$  is path ideal of G. Then

$$Bight(I_2(G)) \leq pd_2(G).$$

Proof. Let *P* be a minimal vertex cover with maximal cardinality. Then by Proposition 5, *P* is an associated prime of  $R/I_2(G)$ , so

$$pd_2(G) = pd(R/I_2(G)) \ge pd_{R_p}(R_p/I_2(G)R_p) = dimR_p = htP.$$

**Proposition 19.** Let  $K_n$  denote the complete graph on *n* vertices and let  $K_{m,n}$  denote the complete bipartite graph on m + n vertices.

- a)  $pd_2(K_n) = n 2$ b)  $pd_2(K_{m,n}) = m + n 2$ .

Proof. a) The proof is by induction on *n*. If n = 2 or 3, the result easy follows. Let  $n \ge 4$  and suppose that for every complete graphs  $K_n$  of other less than n the result is true. Since  $Bight(I_2(K_n)) = n - 2$ then by Proposition  $pd_2(K_n) \ge n-2$ . On the other hand by the inductive hypothesis, we have  $pd_2(K_{n-1}) = n - 3$ , so by Proposition 17

$$pd_2(K_n) \le max\{n-2, n-2\}$$

this completes the proof.

b) Again we use by induction on m + n. If n + m = 2 or 3, then it is easy to see that  $pd_2(K_{m,n}) = 0$  or 1. Let  $n + m \ge 4$  and suppose that for every complete bipartite graph  $K_{m,n}$  of order less than m + n the result is true. Since  $Bight(I_2(K_{m,n})) = m + n - 2$  then  $pd_2(K_{m,n}) \ge m + n - 2$ . Also, by the inductive hypothesis we have  $pd_2(K_{m-1,n}) = m + n - 3$  and  $pd_2(K_{m,n-1}) = m + n - 3$ . So by Proposition 17,  $pd_2(K_{m,n}) \le max\{m + n - 2, pd_2(K_{m-1,n}) + 1, pd_2(K_{m,n-1}) + 1\} = m + n - 2.$ 

As desired.

Corollary 20. Let  $S_n$  denote the star graph on n + 1 vertices and  $S_{m,n}$  denote the double star, then  $pd_2(S_{m,n}) = m + n.$ 

Proof. It follows from Proposition 19, with assume m = 1 and it is easy to see that Bight  $I_2(S_{m,n}) = m + n$  and so by Proposition 17, it follows that  $pd_2(S_{m,n}) = m + n$ .



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