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A Review of The Disteributional Derivatives in The Littlewood-paley Inequalities

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Abstract

We known that from the univariate wavelet ψ, we can construct efficient bases for $L_2(\mathbb{R})$ and other function spaces by dilation and shifts. Also using Given a univariate function ψ, we can obtain a multivariate family of functions by taking tensor products. In this paper by using Littlewood-paley inequalities we will make for our approach the some assumptions a bout the multivariate basis and its dual basis.We study the wavelet bases formed by tensor products of univariate wavelets. From that the Littlewood-paley theory applies to many other orthogonal and nonorthogonal expansions, in this paper first by using the Littlewood-paley inequalities we stated the assumptions a bout multivariate basis. Then under concideration univariate wavelet we define the seminorm for the set of all functions in $L_p(\mathbb{R}^d)$ whose distributional derivatives are in $L_p(\mathbb{R}^d)$ space, also in format a theorem we acquired Necessary and sufficient condition to belong members of $L_p(\mathbb{R}^d)$ to mentioned space.

Keywords: Littlewood-paley inequalities, hyperbolic wavelet, univariate function, tensor product, dyadic rectangles.

1. Introduction

Some of the functions form a stable basis(orthogonal basis in the case of an orthogonal wavelet function)for $L_2(\mathbb{R})$. Previously the Jackson and Bernstein inequalities applied for the convergence of fourier series and the derivative of wavelet polynomial, estimated by using that inequalities.

Littlewood-paley theory has along and important history in harmonic analysis. For the most part, we will utilize known aspects of this theory adapted to the case of hyperbolic wavelet decompositions. Littlewood-paley theory gives a way of characterizing norms of linear combinations of certain basis functions. Its roots lie in the Littlewood-paley theorems for fourier series. The theory applies to many other orthogonal and nonorthogonal expansions (Frazier and Jawerth, 1990), (Frazier et al, 1991) or (Meyer, 1990). We begin in this section by introducing various forms of the Littlewood-paley theory for systems of functions.

2. Definitions and Prerequisites

Let ψ be a wavelet function that satisfies multiresolution analysis (Daubechies,1992). From the univariate wavelet ψ, we can construct efficient bases for $L_2(\mathbb{R})$ and other functions spaces by dilation and shifts. For example, the functions

$$\psi_{j,k} := 2^{k/2}\psi(2^k.-j), \quad j,k \in \mathbb{Z}, \tag{1}$$

Form a stable basis (orthogonal basis in the case of an orthogonal wavelet ψ) for $L_2(\mathbb{R})$.
It is convenient to use adifferent indexing for the functions $\psi_{j,k}$.

Definition 2.1. let $\mathbf{D}(\mathbb{R})$ denote the set of dyadic intervals. Each such interval $\mathbf{I}$ is of the form $\mathbf{I} = [j2^{-k},(j+1)2^{-k}]$. We define

$$\psi_{\mathbf{I}} := \psi_{j,k}, \quad \mathbf{I} = [j2^{-k},(j+1)2^{-k}]. \tag{2}$$

Thus the basis $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is the same as $\{\psi_{\mathbf{I}}\}_{\mathbf{I}\in\mathbf{D}(\mathbb{R})}$.

Definition 2.2. let $\mathbf{D}(\mathbb{R}^d)$ denote the set of dyadic rectangles in $\mathbb{R}^d$. Any $\mathbf{I}\in\mathbf{D}(\mathbb{R}^d)$ is of the form $\mathbf{I} = \mathbf{I}_1 \times \ldots \times \mathbf{I}_d$ with $\mathbf{I}_1,\ldots,\mathbf{I}_d \in \mathbf{D}(\mathbb{R})$. we define

$$\psi_{\mathbf{I}}(x_1,\ldots,x_d) := \psi_{\mathbf{I}_1}(x_1)\ldots\psi_{\mathbf{I}_d}(x_d), \quad \mathbf{I}\in\mathbf{D}(\mathbb{R}^d). \tag{3}$$

Definition 2.3. let r is a positive integer, we denote the differential operator with D^r and define its as follows

$$D^r := \frac{\partial^r}{\partial {x_1}^r}\cdots\frac{\partial^r}{\partial {x_d}^r}. \tag{4}$$

Remark 2.4. For $1\le p \le \infty$, we let $w^r(L_p(\mathbb{R}^d))$ be the set of all functions f in $L_p(\mathbb{R}^d)$ whose distributional derivative $D^r f$ is in $L_p(\mathbb{R}^d)$ and define the seminorm on $w^r(L_p(\mathbb{R}^d))$ by

$$|f|_{w^r(L_p(\mathbb{R}^d))} := \|D^r f\|_p. \tag{5}$$

Remark 2.5. Let $\mathcal{D} = \mathcal{D}(\mathbb{R}^d)$ denote the collection of all dyadic rectangles in $\mathbb{R}^d$. Thus, a rectangle $\mathbf{I} \subset \mathbb{R}^d$ is in $\mathcal{D}$ if and only if $\mathbf{I} = \mathbf{I}_1 \times \dots \times \mathbf{I}_d$ with $\mathbf{I}_1, \dots, \mathbf{I}_d$ dyadic intervals in $\mathbb{R}$. In the univariate case, on particular way to obtain such systems in by shifts and dilates of a univariate wavelet.
For a univariate function ψ, we use notation $\psi_{\mathbf{I}}$ of (2) to denote its $\boldsymbol{L_2}(\mathbb{R})$ –normalized,shifted dilates. The function ψ is an orthogonal wavelet if the collection of functions $\psi_{\mathbf{I}}$, $\mathbf{I}\epsilon\mathcal{D}(\mathbb{R})$, forms a complet orthonormal system for $\boldsymbol{L_2}(\mathbb{R})$. Other cases of interest in wavelet theory are the prewavelets (De Boor, et al, 1993), spline wavelets (Chui and Wang, 1992), and biorthogonal wavelets (Cohen et al, 1992).
In the latter cases, the orthogonality of the family $\psi_{\mathbf{I}}$, $\mathbf{I}\epsilon\mathcal{D}(\mathbb{R})$, is replaced by $\boldsymbol{L_2} - \text{stability}$ (Riesz basis).
Given a univariate function ψ, we can obtain a multivariate family of functions by taking tensor products. For rectangles $\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, we define

$$\psi_{\mathbf{I}}(x_1, \dots, x_d) := \psi_{\mathbf{I}_1}(x_1) \dots \psi_{\mathbf{I}_d}(x_d), \quad \mathbf{I} = \mathbf{I}_1 \times \dots \times \mathbf{I}_d. \tag{6}$$

We will use the notation $\psi_{\mathbf{I}}$ to denote the family of functions obtained by tensor products of shifted dilates of univariate function ψ.

3. Application of the Littlewood-paley inequalities

We will give important approach to proving companion Jackson and Bernstein inequalities in this section. This approach assumes certain conditions on ψ that allow us to characterize $\boldsymbol{w}^r(\boldsymbol{L_p}(\mathbb{R}^d))$, $1 < p < \infty$, interms of expansion in the basis $\psi_{\mathbf{I}}$, $\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$. Using this characterization, then we can easily prove the Jackson and Bernstein inequalities.

Defintion 3.1. Let $\psi_{\mathbf{I}}$, $\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, is a schauder basis, then associated to this basis we have its dual basis. In the case that $1 < p < \infty$, the dual basis is given by linear functionals $c_{\mathbf{I}}$ with

$$c_{\mathbf{I}}(f) = \int_{\mathbb{R}^d} f(x)\lambda(\mathbf{I}, x)\, dx \tag{7}$$

And the functions $\lambda(\mathbf{I}, .)$ are in $\boldsymbol{L_{p'}}(\mathbb{R}^d)$ with $1/_p + 1/_{p'} = 1$.

If we set $\lambda(x) := \lambda([\mathbf{0,1}], x)$, then it is easy to see (by using shifts and dilations) that we can take $\lambda(\mathbf{I}, .) = \lambda_{\mathbf{I}}$, $\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, with the $\lambda_{\mathbf{I}}$ defined as in (6). we note that in the case that ψ is suitably differentiable, we have $(D^r\psi)_{\mathbf{I}} = |\mathbf{I}|^r D^r \psi_{\mathbf{I}}, \mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$. we will make for our approach the following assumptions about the multivariate basis $\psi_{\mathbf{I}}, \mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, and its dual basis $\lambda_{\mathbf{I}}$, $\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$:

(A_1) $\psi_{\mathbf{I}}, \mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, span $\boldsymbol{L_p}(\mathbb{R}^d)$, $1 < p < \infty$, and satisfy the Littlewood-paley inequalities;
(A_2) $(D^r\psi)_{\mathbf{I}}, \mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, span $\boldsymbol{L_p}(\mathbb{R}^d)$, $1 < p < \infty$, and satisfy the Littlewood-paley inequalityes ;
(A_3) $\int_{\mathbb{R}} x^j\, \lambda(x)\, dx = 0$, $\quad$ j = 0,...,r ; and
(A_4) $|\lambda(x)| \leq C.\max(1, |x|^{-r-1-\varepsilon})$, for some $\varepsilon > 0$.

Because of (A_3) and (A_4), we can integrate the univareate function λ, r times to find a function $\mu \epsilon \boldsymbol{L_{p'}}(\mathbb{R})$ which satisfies $(-1)^r \mu^{(r)} = \lambda$. It follows that $D^r\mu_{\mathbf{I}} = (-1)^{rd}|\mathbf{I}|^{-r}\lambda_{\mathbf{I}}$ $, \mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$. integration by parts then shows that

$$\int_{\mathbb{R}^d} (D^r\psi)_{\mathbf{I}}\,\mu_{\mathbf{J}}\,\mathrm{dx} = |\mathbf{I}|^r|\mathbf{J}|^{-r}\int_{\mathbb{R}^d} \psi_{\mathbf{I}}\,\lambda_{\mathbf{J}}\,\mathrm{dx} = \delta(\mathbf{I},\mathbf{J}), \quad \mathbf{I},\mathbf{J}\epsilon\mathcal{D}(\mathbb{R}^d), \tag{8}$$

with δ the Kronecker delta. Hence $\mu_{\mathbf{I}},\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, is the dual basis for $(D^r\psi)_{\mathbf{I}}$, $\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$.

Theorem 3.2. let r be a positive integer, $1 < p < \infty$, and let ψ be a univariate function which satisfies assumptions (A_1)-(A_4). Then a function f ϵ $\boldsymbol{L_p}(\mathbb{R}^d)$ is in $\boldsymbol{w}^r(\boldsymbol{L_p}(\mathbb{R}^d))$ if and only if

$$\mathrm{f} = \sum_{\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)} \mathrm{c}_{\mathbf{I}}(f)\,\psi_{\mathbf{I}} \tag{9}$$

With

$$\left\| \left(\sum_{\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)} [|\mathbf{I}|^{-r}|\mathrm{c}_{\mathbf{I}}(f)|\chi_{\mathbf{I}}]^2\right)^{\frac{1}{2}}\right\|_p < \infty. \tag{10}$$

Furthermore, the left side of (10) is equivalent to $|f|_{\boldsymbol{w}^r(\boldsymbol{L_p}(\mathbb{R}^d))}$.

Proof. Suppose first that f ϵ $\boldsymbol{w}^r\left(\boldsymbol{L_p}(\mathbb{R}^d)\right)$. Assumption (A_1) gives that the functions $\psi_{\mathbf{I}},\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, satisfy the strong Littlewood-paley inequalities. Hence, these functions are an unconditional basis for $\boldsymbol{L_p}(\mathbb{R}^d)$ and $\mathrm{f} = \sum_{\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)} \mathrm{c}_{\mathbf{I}}(f)\,\psi_{\mathbf{I}}$
with

$$\mathrm{c}_{\mathbf{I}}(f) = \int_{\mathbb{R}^d} f\,\lambda_{\mathbf{I}}\,\mathrm{dx}. \tag{11}$$

Likewise, the functions $(D^r\psi)_{\mathbf{I}},\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, are also a basis for $\boldsymbol{L_p}(\mathbb{R}^d)$, and we have $D^r\mathrm{f} = \sum_{\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)} d_{\mathbf{I}}(f)(D^r\psi)_{\mathbf{I}}$ with

$$d_{\mathbf{I}}(f) = \int_{\mathbb{R}^d} D^r f\,\mu_{\mathbf{I}}\mathrm{dx} = (-1)^{rd}\int_{\mathbb{R}^d} f\,D^r\mu_{\mathbf{I}}\mathrm{dx} = |\mathbf{I}|^{-r}\int_{\mathbb{R}^d} f\,\lambda_{\mathbf{I}}\,\mathrm{dx} = |\mathbf{I}|^{-r}\mathrm{c}_{\mathbf{I}}(f). \tag{12}$$

We can compute $\|D^r\mathrm{f}\|_p$ from the Littlewood-paley condition for the basis $(D^r\psi)_{\mathbf{I}},\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$.
This gives that the left side of (10) is equivalent to $\|D^r\mathrm{f}\|_p$.

Conversely, assume that f ϵ $\boldsymbol{L_p}(\mathbb{R}^d)$ is such that (10) is finite. Because $\psi_{\mathbf{I}},\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, is an unconditional basis for $\boldsymbol{L_p}(\mathbb{R}^d)$, we have

$$\mathrm{f} = \sum_{\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)} \mathrm{c}_{\mathbf{I}}(f)\,\psi_{\mathbf{I}}$$

in the sense of $\boldsymbol{L_p}(\mathbb{R}^d)$-convergence. From (10) and the fact that $(D^r\psi)_{\mathbf{I}},\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$, satisfies the Littlewood-paley inequalities, we find that there is a function g ϵ $\boldsymbol{L_p}(\mathbb{R}^d)$ with

$$\mathrm{g} = \sum_{\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)} |\mathbf{I}|^{-r}\mathrm{c}_{\mathbf{I}}(f)\,(D^r\psi)_{\mathbf{I}} \tag{13}$$

again in the sense of $\boldsymbol{L_p}(\mathbb{R}^d)$-convergence. We compute the coefficients of g with respect to the basis $(D^r\psi)_{\mathbf{I}},\mathbf{I}\epsilon\mathcal{D}(\mathbb{R}^d)$ and find

$$\int_{\mathbb{R}^d} g\,\mu_{\mathbf{I}} = |\mathbf{I}|^{-r}\mathrm{c}_{\mathbf{I}}(f) = |\mathbf{I}|^{-r}\int_{\mathbb{R}^d} f\,\lambda_{\mathbf{I}} = (-1)^{rd}\int_{\mathbb{R}^d} f\,D^r\mu_{\mathbf{I}}. \tag{14}$$

this shows that g is the disteributional derivative and therefor f $\epsilon\, w^r\left(L_p(\mathbb{R}^d)\right)$. ∎

4. Conclusion

In this paper, after presenting definitions and prerequisites, first by using the Littlewood-paley inequalities we stated the assumptions a bout multivariate basis $\psi_I, I\epsilon \mathcal{D}(\mathbb{R}^d)$. Also under concideration univariate wavelet we defin e the seminorm for $w^r\left(L_p(\mathbb{R}^d)\right)$ space, then we acquired the Necessary and sufficient condition to belong members of $L_p(\mathbb{R}^d)$ to $w^r\left(L_p(\mathbb{R}^d)\right)$ space.

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