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Solutions of Eckart potential plus Hulthen potential in the Presence of Spin Symmetry and Pseudo spin Symmetry

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Abstract

In this paper we solve analytically Dirac equation for Eckart potential plus Hulthen potential with Spin Symmetry and Pseudo spin Symmetry for $k\neq0$. The Parametric Nikiforov-Uvarov (PNU) method is used to obtain the energy Eigen-values and wave functions. We also discuss the energy Eigen-values and wave functions for the special case of modified Eckart plus Hulthen potential for the spin and pseudo spin symmetry with PNU method**.** To show the accuracy of the present model, some numerical values of the energy levels are shown.

Keywords: Dirac equation; Eckart potential; Hulthen potential, Nikiforov-Uvarov (NU) method; Spin and Pseudo spin symmetry

Introduction

In quantum mechanics, it is well known that the exact solutions play fundamental role, because, this solutions usually contain all the necessary information about the quantum mechanical model under investigation (Arima et al., 1969). Therefore one of the interesting problems in nuclear and high energy physics is to obtain exact solution of the Klein - Gordon, Duffin – Kemmer - Petiau and Dirac equations for mixed vector and scalar potentials (Oyewumi et al., 2010). The study of relativistic effects is always useful in some quantum mechanical systems (Wang et al., 1988). Therefore, the Dirac equation has become the most appealing relativistic wave equation for spin-1/2 particles. For example, in the relativistic treatment of nuclear phenomena the Dirac equation is used to describe the behavior of the nuclei in nucleus and also in solving many problems of high-energy physics and chemistry. For this reason, it has been used extensively to study the relativistic heavy ion collisions, heavy ion spectroscopy and more recently in laser–matter interaction (for a review, see (Yousef et al., 2006) and references therein) and condensed matter physics (Cheng et al., 2007).

The idea about spin symmetry and pseudo-spin symmetry with the nuclear shell model has been introduced in 1969 by Arima et al. (1969), Hecht and Adler (1969) (Arima et al., 1969). Spin and pseudo spin symmetries are SU(2) symmetries of a Dirac Hamiltonian with vector and scalar potentials. They are realized when the difference, $\Delta(r)=V(r)-S(r)$, or the sum, $\Sigma(r)=V(r)+S(r)$, are constants. The near realization of these symmetries may explain degeneracy in some heavy meson spectra (spin symmetry) or in single-particle energy levels in nuclei (pseudo spin symmetry), when these physical systems are described by relativistic mean-field theories (RMF) with scalar and vector potentials (Alberto et al., 2013). The kind of various methods have been used for the exact solutions of the Klein–Gordon equation and Dirac equation such as the Super symmetric Quantum Mechanics(Feizi et al. 2011), Asymptotic iteration method(AIM) (Ciftci et al., 2003), factorization method (Infeld et al., 1951), Laplace transform approach (Arda et al., 2010) ,GPS METHOD (Amlan, 2004) and the path integral method (Diaf et al., 2011), Nikiforov-Uvarov(NU) (Shojaei et al., 2014) and others. The Klein – Gordon and Dirac wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics with some typical potential by using different methods (Ikot et al., 2011). For example, Kratzer potential(Qiang, 2004), Woods-Saxon potential(Guo et al., 2005), Scarf potential(Zhang et al., 2005), Hartmann potential(Chen, 2005), Rosen Morse potential(Alhaidari, 2001), Hulth'en potential(Farrokh et al., 2013) and Eckart potential (Babatunde, 2012).

In this paper, we attempt to solve approximately Dirac wave equation for $k\neq 0$ with Eckart plus Hulthen potential for the spin and pseudo spin symmetry by using the Nikiforov – Uvarov (NU) method. We also discuss the special case of modified Eckart plus Hulthen potential for the spin and pseudo spin symmetry with PNU method. The organization of this paper is as follows: in Section 2, the PNU method is reviewed . In section 3 we review Basic Dirac Equations briefly. In section 4.1 and 4.2, solutions of Dirac wave equation for the spin symmetry and pseudo spin symmetry of this potential is presented, respectively. Section 5 contains discussion of some special cases for modified Eckart plus Hulthen potential by the spin and pseudo spin symmetry. The conclusion is given in Section 6.

2 REVIEW OF PARAMETRIC NIKIFOROV -UVAROV (PNU) METHOD

The NU method is based on the solution of a generalized second order differential linear equation with special orthogonal functions. The NU method has been used to solve the Schr¨odinger, Dirac, and Klein-Gordon wave equations for a certain kind of potential**.** In this method the differential equations can be written as follows (Shojaei et al., 2015):

$$\Psi''(s)+\frac{\tilde{\tau}(s)}{\sigma(s)}\Psi'(s)+\frac{\tilde{\sigma}(s)}{\sigma^2(s)}\Psi(s)=0 \quad (1)$$

Where $\sigma(s)$ and $\tilde{\sigma}(s)$ are polynomials at most second degree and $\tilde{\tau}(s)$ is a first degree polynomials.

To make the application of the NU method simpler and direct without need to check the validity of solution. We present a shortcut for the method. Hence, firstly, we write the general form of the Schrödinger-like equation (Eq. (1)) in a rather more general form as (Babatunde, 2013)

$$\left[\frac{d^2}{ds^2}+\frac{c_1-c_2s}{s(1-c_3s)}\frac{d}{ds}+\frac{(-p_2s^2+p_1s-p_0)}{s^2(1-c_3s)^2}\right]\Psi_n(s)=0 \quad (2)$$

We set the wave function as**:**

$$\Psi(s)=\phi(s)y(s) \quad (3)$$

Secondly, we compare Eq. (2) with its counterpart Eq. (1) to obtain the following parameter values,

$$\tilde{\tau}(s)=c_1-c_2s\,,\ \sigma(s)=s(1-c_3s)\,,\ \tilde{\sigma}(s)=-p_2s^2+p_1s-p_0 \quad (4)$$

Now, following the NU method, we obtain the following energy equation (43):

$$nc_2-(2n+1)c_5+(2n+1)(\sqrt{c_9}+c_3\sqrt{c_8})+n(n-1)c_3+c_7+2c_3c_8+2\sqrt{c_8c_9}=0 \quad (5)$$

And the corresponding wave functions

$$\rho(s)=s^{c_{10}}(1-c_3s)^{c_{11}}\,,\Phi(s)=s^{c_{12}}(1-c_3s)^{c_{13}}\quad c_{12}\rangle 0, c_{13}\rangle 0 \quad (6)$$

$$y_n(s)=P_n^{(c_{10},c_{11})}(1-2c_3s), c_{10}\rangle -1, c_{11}\rangle -1 \quad (7)$$

$$\Psi_{n,k}(s)=N_{n,k}s^{c_{12}}(1-c_3s)^{c_{13}}P_n^{(c_{10},c_{11})}(1-2c_3s) \quad (8)$$

Where $P_n^{(\mu,\nu)}(x)$ ($\mu>-1$, $v>-1$ and $x\ \varepsilon\ (-1, 1)$) are Jacobi polynomials with the following constants:

$$c_4=\frac{1}{2}(1-c_1) \qquad c_5=\frac{1}{2}(c_2-2c_3)$$

$$c_6=c_5^2+p_2 \qquad c_7=2c_4c_5-p_1$$

$$c_8=c_4^2+p_0 \qquad c_9=c_3(c_7+c_3c_8)+c_6$$

$$c_{10}=c_1+2c_4+2\sqrt{c_8}-1\rangle -1 \qquad c_{11}=1-c_1-2c_4+\frac{2}{c_3}\sqrt{c_9}\rangle -1, c_3\neq 0$$

$$c_{12}=c_4+\sqrt{c_8}\rangle 0 \qquad c_{13}=-c_4+\frac{1}{c_3}\left(\sqrt{c_9}-c_5\right)\rangle 0, c_3\neq 0 \quad (9)$$

Where $c_{12}>0$, $c_{13}>0$ and $s \in (0, 1/c_3)$, $c_3\neq 0$, In the rather more special case of $c_3=0$, c_{11}, c_{13} becomes

$$c_{11}=c_2-2c_5+2(\sqrt{c_9}+c_3\sqrt{c_8} \qquad c_{13}=c_5-(\sqrt{c_9}-c_3\sqrt{c_8}) \quad (10)$$

The wave function (Eq.8) becomes

$$\lim_{c_3\to 0}P_n^{(c_{10},c_{11})}(1-2c_3s)=L_n^{c_{10}}(c_{11}s) \quad (11)$$

$$\lim_{c_3\to 0}(1-2c_3s)^{c_{12}}=e^{c_{13}s} \quad (12)$$

$$\Psi_{n,k}(s)=N_{n,k}s^{c_{12}}e^{c_{13}s}L_n^{c_{10}}(c_{11}s) \quad (13)$$

And $L_n^{\alpha}(x)$ are the Laguerre polynomials (Auvil et al., 1978).

3 Basic Dirac Equations

In the relativistic description, the Dirac equation of a single-nucleon with the mass moving in an attractive scalar potential S(r) and a repulsive vector potential V (r) can be written as (Walter, 2000)

$$[-i\hbar c\hat{\alpha}.\hat{\nabla}+\hat{\beta}(Mc^2+S(r))]\Psi_{n_r,k}=[E-V(r)]\Psi_{n_r,k} \quad (14)$$

Where *E* is the relativistic energy, *M* is the mass of a single particle and α and β are the 4 ×4 Dirac matrices. For a particle in a central field, the total angular momentum J and $\hat{K}=-\hat{\beta}(\hat{\alpha}.\hat{L}+\hbar)$ commute with the Dirac Hamiltonian where L is the orbital angular momentum. For a given total angular momentum j, the Eigen-values of the $\hat{K}$ are k=± (j+1/2) where negative sign is for aligned spin and positive sign is for unaligned spin. The wave-functions can be classified according to their angular momentum *j* and spin-orbit quantum number *k* as follows

$$\Psi_{n_r,k}(r,\theta,\phi)=\frac{1}{r}\begin{bmatrix} F_{n_r,k}(r)Y^{l}_{jm}(\theta,\phi) \\ iG_{n_r,k}(r)Y^{\tilde{l}}_{jm}(\theta,\phi) \end{bmatrix} \tag{15}$$

Where $F_{n_r,k}(r)$ and $G_{n_r,k}(r)$ are upper and lower components, $Y^{l}_{jm}(\theta,\phi)$ and $Y^{\tilde{l}}_{jm}(\theta,\phi)$ are the spherical harmonic functions. n_r is the radial quantum number and m is the projection of the angular momentum on the z axis. The orbital angular momentum quantum numbers l and $\tilde{l}$ represent to the spin and pseudo spin quantum numbers. Substituting Eq. (15) into Eq. (14), we obtain couple equations for the radial part of the Dirac equation as follows

$$\begin{cases} (\frac{d}{dr}+\frac{k}{r})F_{n_r,k}(r)=\frac{1}{\hbar c}[Mc^2+E-\Delta(r)]G_{n_r,k}(r) \\ (\frac{d}{dr}-\frac{k}{r})G_{n_r,k}(r)=\frac{1}{\hbar c}[Mc^2-E+\sum(r)]F_{n_r,k}(r) \end{cases} \tag{16}$$

Where $\Delta(r)=V(r)-S(r)$ and $\sum(r)=V(r)+S(r)$ are the difference and the sum of the potentials V(r) and S(r), respectively. Under the condition of the spin symmetry, i. e. $\Delta(r)=0$, Eq. (16) reduces into

$$(-\frac{d^2}{dr^2}+\frac{k(k+1)}{r^2}+\frac{1}{\hbar^2c^2}[Mc^2+E][Mc^2-E+\sum(r)]F_{n_r,k}(r)=0 \tag{17}$$

Under the condition of the pseudo spin symmetry, i.e., $\sum(r)=0$ Eq. (16) turns to the following form

$$(-\frac{d^2}{dr^2}+\frac{k(k-1)}{r^2}+\frac{1}{\hbar^2c^2}[Mc^2-E][Mc^2+E-\Delta(r)]G_{n_r,k}(r)=0 \tag{18}$$

We consider bound state solutions that demand the radial components satisfying $F_{n_r,k}(0)=G_{n_r,k}(0)=0$, and $F_{n_r,k}(\infty)=G_{n_r,k}(\infty)=0$ (48).

4.1 Solution Dirac Equations for Spin Symmetric

Under the condition of the spin symmetry, i. e. $\Delta(r)=0$, the Dirac equation can be written as

$$(-\frac{d^2}{dr^2}+\frac{k(k+1)}{r^2}+\frac{1}{\hbar^2c^2}[Mc^2+E][Mc^2-E+\sum(r)]F_{n_r,k}(r)=0 \tag{19}$$

The Eckart potential (40, 41) plus Hulthen potential (39) is defined as

$$V(r)=q_1\cos ech^2(\alpha r)-q_2\coth(\alpha r)+\frac{v_0}{(1-e^{-2\alpha r})}-\frac{v_1}{(1-e^{-2\alpha r})^2} \tag{20}$$

And can be rewritten in the exponential form as

$$V(r)=4q_1\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}-q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})}+\frac{v_0}{(1-e^{-2\alpha r})}-\frac{v_1}{(1-e^{-2\alpha r})^2} \tag{21}$$

Where the parameters q_1, q_2, v_0 and v_1 are positive and real parameters, these parameters describe the depth of the potential well, and the parameter α is related to the range of the potential. In fig. 1, we show the behavior of the Eckart potential plus Hulthen potential as a function of r for three screening parameter values α = 0.12, 0.25, 0.45 fm^{-1} by taking the strength parameters $q_1= v_0= 0.8$, $q_2= v_1=2$. It is seen that the potential strength decreases with the increasing of the screening parameter value.

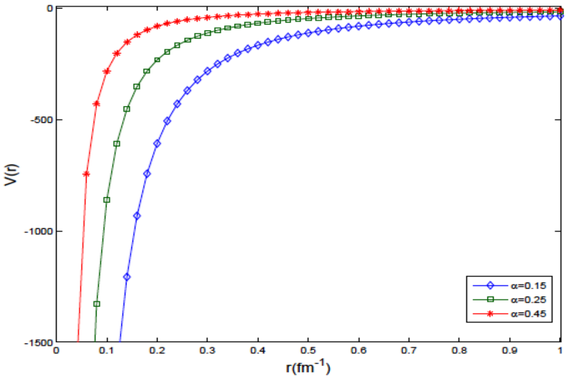

Fig. 1. The behavior of the Eckart potential plus Hulthen potential.

Under the condition of the spin symmetry sum of the potentials V (r) and S(r) can be written as

$$\Sigma(r)=8q_1\frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2}-2q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})}+\frac{2v_0}{(1-e^{-2\alpha r})}-\frac{2v_1}{(1-e^{-2\alpha r})^2} \qquad (22)$$

If we define a new variable $F_{n_r,k}(r)=rR(r)$ and substituting it in to Eq. (19), we obtain the radial equation of Dirac equation as

$$(\frac{d^2}{dr^2}+\frac{2}{r}+\frac{(E^2-M^2c^4)}{\hbar^2c^2}-\frac{(E+Mc^2)}{\hbar^2c^2}\Sigma(r)-\frac{k(k+1)}{r^2}]R_{n_r,k}(r)=0 \qquad (23)$$

We can evaluate the new improved approximation scheme by using the following pekeris-type approximation that is valid for $\alpha\leq 1$, (49)

$$\frac{1}{r^2}\approx\frac{4\alpha^2}{(e^{-2\alpha r}-1)^2} \qquad (24)$$

Using the transformation $s=(1-\exp(-2\alpha r))$ Eq. (23) brings into the form

$$R''(s)+\frac{(-4\alpha)}{s}R'(s)+\{\frac{(E^2-M^2c^4)}{\hbar^2c^2}-\frac{(E+Mc^2)}{\hbar^2c^2}[8q_1\frac{(1-s)}{s^2}-2q_2\frac{(2-s)}{s}+\frac{2v_0}{s}-\frac{2v_1}{s^2}]$$
$$-\frac{4\alpha^2k(k+1)}{s^2}\}R_{n_r,k}(r)=0 \qquad (25)$$

We can write the Eq. (25) as summarized below

$$R''(s)+\frac{(-4\alpha)}{s}R'(s)+\frac{1}{s^2}\{\frac{(E^2-M^2c^4)-2q_2(E+Mc^2)}{\hbar^2c^2}s^2+\frac{(E+Mc^2)(8q_1+4q_2-2v_0)}{\hbar^2c^2}s$$
$$-\frac{(E+Mc^2)(8q_1-2v_1)}{\hbar^2c^2}-4\alpha^2k(k+1)\}R(s)=0 \qquad (26)$$

We can write the Eq. (26) as summarized below

$$R''(s)+\frac{(-4\alpha)}{s}R'(s)+\frac{1}{s^2}[-p_2s^2+p_1s-p_0]R(s)=0 \qquad (27)$$

Where theparametersp$_2$, p1 and p$_0$areconsideredas follows:

$$p_2=\frac{(E+Mc^2)}{\hbar^2c^2}[2q_2-(E-Mc^2)]$$
$$p_1=\frac{(E+Mc^2)}{\hbar^2c^2}[8q_1+4q_2-2v_0] \qquad (28)$$
$$p_0=\frac{(E+Mc^2)}{\hbar^2c^2}[8q_1-2v_1]+4\alpha^2k(k+1)$$

Comparing Eq. (27) with Eq. (2), we can easily obtain the coefficients c_i (i = 1, 2, 3) as follows:

$$C_1=-4\alpha,\ C_2=C_3=0 \tag{29}$$

The values of the coefficients c_i (i = 4, 5, …13) are also found from Eq.(9) and Eq.(10) as below:

$$c_4=\frac{(4\alpha+1)}{2} \qquad c_5=0$$
$$c_6=c_9=p_2 \qquad c_7=p_1$$
$$c_8=p_0+\frac{(4\alpha+1)^2}{4} \qquad c_{10}=2\sqrt{p_0+\frac{(4\alpha+1)^2}{4}}$$
$$c_{11}=2\sqrt{p_2} \qquad c_{12}=\frac{(4\alpha+1)}{2}+\sqrt{p_0+\frac{(4\alpha+1)^2}{4}}$$
$$c_{13}=-\sqrt{p_2} \tag{30}$$

Using the energy equation, Eq. (5) for energy Eigen-values we have:

$$(2n+1)\sqrt{p_2}-p_1+2\sqrt{p_2[p_0+\frac{(4\alpha+1)^2}{4}]}=0 \tag{31}$$

And using Eq. (28) we can obtain the energy Eigen-values equation, in closed form, as:

$$[2q_2-(E-Mc^2)]\left[(2n+1)+2\sqrt{\frac{(E+Mc^2)}{\hbar^2c^2}[8q_1-2v_1]+4\alpha^2k(k+1)+\frac{(4\alpha+1)^2}{4}}\right]^2 =\frac{(E+Mc^2)}{\hbar^2c^2}[8q_1+4q_2-2v_0]^2 \tag{32}$$

In Fig. 2, Under the condition of the spin symmetry we show the behavior of the energy Eigen-values equation for three levels energy ($E_{1,0}$, $E_{2,1}$ and $E_{3,2}$) for some of screening parameter values (α) by taking the strength parameters $q_1=q_2=v_0=v_1$ 0.8. It is seen that the energy Eigen-values increase with the increasing of the screening parameter value.

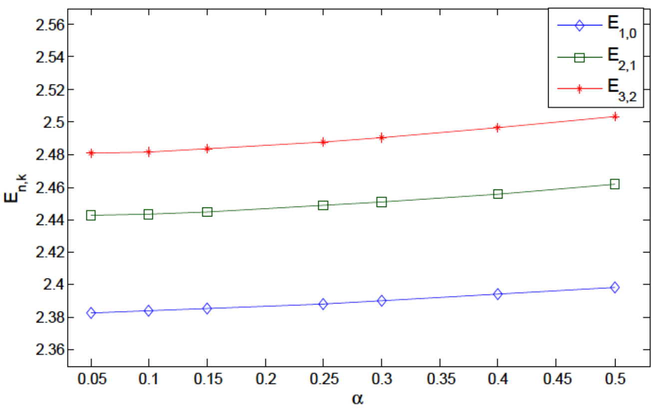

Fig.2.The behavior of the energy Eigen-values for some of screening parameter values (α).

Let us find the corresponding wave functions. In reference to Eq. (6), Eq. (7) and using Eq. (30), we find the following useful functions

$$\phi(s)=s^{[\frac{(4\alpha+1)}{2}+(p_0+\frac{(4\alpha+1)^2}{4})^{\frac{1}{2}}]}e^{-(p_2)^{\frac{1}{2}}s} \tag{33}$$

$$y_n(s)=L_n^{2\sqrt{p_0+\frac{(4\alpha+1)^2}{4}}}(2\sqrt{p_2}s) \tag{34}$$

Thus, using the relation Eq. (13), we can obtain the wave function with the aid of Eq. (30) as

$$R(s)=B_n s^{[\frac{(4\alpha+1)}{2}+\sqrt{p_0+\frac{(4\alpha+1)^2}{4}}]} e^{-\sqrt{p_2}s} L_n^{2\sqrt{p_0+\frac{(4\alpha+1)^2}{4}}}(2\sqrt{p_2}s) \quad (35)$$

By using $F_{n_r,k}(r)=rR(r)$ we find finally wave function

$$F_{n_r,k}(r)=Nr(1-e^{-2\alpha r})^{[\frac{(4\alpha+1)}{2}+\sqrt{p_0+\frac{(4\alpha+1)^2}{4}}]}\exp[-\sqrt{p_2}(1-e^{-2\alpha r})]L_n^{2\sqrt{p_0+\frac{(4\alpha+1)^2}{4}}}[2\sqrt{p_2}(1-e^{-2\alpha r})] \quad (36)$$

Where N is the normalization constant, on the other hand, the lower component of the Dirac spinor can be calculated from Eq. (37) as

$$G_{n_r,k}(r)=\frac{\hbar^2c^2}{E+Mc^2}(\frac{d}{dr}+\frac{k}{r})F_{n_r,k}(r) \quad (37)$$

We have obtained the energy Eigen-values and the wave function of the radial Dirac equation for Eckart plus Hulthen potential with the spin symmetry for k≠0**.**

4.2 Solution Dirac Equations for Pseudo spin Symmetric

For the pseudo spin symmetry, i.e., $\sum(r)=0$ the Dirac equation can be written as

$$(-\frac{d^2}{dr^2}+\frac{k(k-1)}{r^2}+\frac{1}{\hbar^2c^2}[Mc^2-E][Mc^2+E-\Delta(r)]G_{n_r,k}(r)=0 \quad (38)$$

If we define a new variable $G_{n_r,k}(r)=rg(r)$ and substituting it in to Eq. (38), we obtain the radial equation of Dirac equation as

$$(\frac{d^2}{dr^2}+\frac{2}{r}+\frac{(E^2-M^2c^4)}{\hbar^2c^2}-\frac{(E-Mc^2)}{\hbar^2c^2}\Delta(r)-\frac{k(k-1)}{r^2}]g_{n_r,k}(r)=0 \quad (39)$$

By using the following pekeris-type approximation and using the transformation $s=(1-e(-2\alpha r))$ we can write the Eq. (39) as summarized below

$$g''(s)+\frac{(-4\alpha)}{s}g'(s)+\frac{1}{s^2}[-p'_2s^2+p'_1s-p'_0]g(s)=0 \quad (40)$$

Where the parameters p'_2, p'_1 and p'_0 are considered as follows:

$$p'_2=\frac{(E-Mc^2)}{\hbar^2c^2}[2q_2-(E+Mc^2)]$$

$$p'_1=\frac{(E-Mc^2)}{\hbar^2c^2}[8q_1+4q_2-2v_0] \quad (41)$$

$$p'_0=\frac{(E-Mc^2)}{\hbar^2c^2}[8q_1-2v_1]+4\alpha^2k(k-1)$$

Comparing Eq. (40) with Eq. (2), we can easily obtain the coefficients c_i (i = 1, 2, 3) as follows:

$C_1=-4\alpha$, $C_2=C_3=0$ (42)

The values of the coefficients c_i (i = 4, 5, …13) are also found from Eq.(9) and Eq.(10) as below

$$c_4=\frac{(4\alpha+1)}{2} \qquad c_5=0 \qquad c_6=c_9=p'_2 \qquad c_7=p'_1$$

$$c_8=p'_0+\frac{(4\alpha+1)^2}{4} \qquad c_{10}=2\sqrt{p'_0+\frac{(4\alpha+1)^2}{4}}$$

$$c_{11}=2\sqrt{p'_2} \qquad c_{12}=\frac{(4\alpha+1)}{2}+\sqrt{p'_0+\frac{(4\alpha+1)^2}{4}}$$

$$c_{13}=-\sqrt{p'_2} \quad (43)$$

Using the energy equation, Eq. (5) for energy Eigen-values we have:

$$(2n+1)\sqrt{p_2'}-p_1'+2\sqrt{p_2'[p_0'+\frac{(4\alpha+1)^2}{4}]}=0 \tag{44}$$

And using Eq. (41) we can obtain the energy Eigen-values equation, in closed form, as:

$$[2q_2-(E+Mc^2)]\left[(2n+1)+2\sqrt{\frac{(E-Mc^2)}{\hbar^2c^2}[8q_1-2v_1]+4\alpha^2k(k-1)+\frac{(4\alpha+1)^2}{4}}\right]^2 = \frac{(E-Mc^2)}{\hbar^2c^2}[8q_1+4q_2-2v_0]^2 \tag{45}$$

In Fig.3, For the pseudo spin symmetry we show the behavior of the energy Eigen-values equation for three levels energy ($E_{1,0}$, $E_{2,1}$ and $E_{3,2}$) for some of screening parameter values (α) by taking the strength parameters $q_1=q_2=v_0=v_1$ 0.8. It is seen that the energy Eigen-values decreases with the increasing of the screening parameter value.

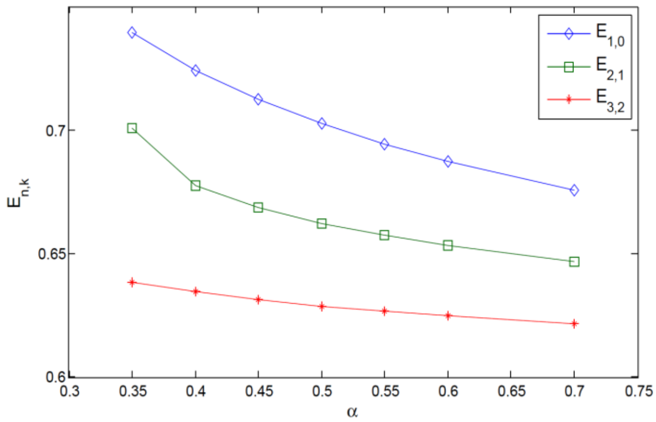

Fig.3 The behavior of the energy Eigen-values for some of screening parameter values (α)

Using Eq. (13) and Eq. (43) we can finally obtain the wave functions as below

$$G_{n_r,k}(r)=Nr(1-e^{-2\alpha r})^{[\frac{(4\alpha+1)}{2}+\sqrt{p_0'+\frac{(4\alpha+1)^2}{4}}]}\exp[-\sqrt{p_2'}(1-e^{-2\alpha r})]L_n^{2\sqrt{p_0'+\frac{(4\alpha+1)^2}{4}}}[2\sqrt{p_2'}(1-e^{-2\alpha r})] \tag{46}$$

Where Nis the normalization constant, on the other hand, the lower component of the Dirac spinor can be calculated by Eq. (47) as

$$F_{n_r,k}(r)=\frac{\hbar^2c^2}{Mc^2-E}(\frac{d}{dr}-\frac{k}{r})G_{n_r,k}(r) \tag{47}$$

Different between spin symmetry and pseudo spin symmetry can be written as:

$$V(r)=S(r)\Leftrightarrow V(r)=-S(r)$$
$$\begin{cases}(E+Mc^2)\Leftrightarrow(E-Mc^2)\\ k(k+1)\Leftrightarrow k(k-1)\end{cases} \tag{48}$$

We have obtained the energy Eigen-values and the wave function of the radial Dirac equation for Eckart plus Hulthen potential with the pseudo spin symmetry for k≠0.

5. Some Special Cases

In this section we consider some special cases of interest If we consider q_1, v_1=0 spatially modified Eckart plus Hulthen potential can be written:

$$V(r)=-q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})}+\frac{v_0}{(1-e^{-2\alpha r})} \tag{49}$$

For spin symmetry the Dirac equation can be written as:

$$\left(-\frac{d^2}{dr^2}+\frac{k(K+1)}{r^2}+\frac{1}{\hbar^2c^2}[Mc^2+E][Mc^2-E+\Sigma(r)]F_{n_r,k}(r)=0 \tag{50}$$

And,

$$\Sigma(r)=-2q_2\frac{(1+e^{-2\alpha r})}{(1-e^{-2\alpha r})}+\frac{2v_0}{(1-e^{-2\alpha r})} \tag{51}$$

By using the following pekeris-type approximation and using the transformation $s=\left(1-\exp\left(-2\alpha r\right)\right)$ and $F_{n_r,k}(r)=rR(r)$ we have:

$$R''(s)+\frac{(-4\alpha)}{s}R'(s)+\frac{1}{s^2}\{\frac{(E^2-M^2c^4)-2q_2(E+Mc^2)}{\hbar^2c^2}s^2+\frac{(E+Mc^2)(2v_0-4q_2)}{\hbar^2c^2}s-4\alpha^2k(k+1)\}R(s)=0 \tag{52}$$

We can write the Eq. (52) as summarized below

$$R''(s)+\frac{(-4\alpha)}{s}R'(s)+\frac{1}{s^2}[-p''_2s^2+p''_1s-p''_0]R(s)=0 \tag{53}$$

Therefore by using (PNU) method we obtain energy Eigen-values equation follow as:

$$p''_2\left[(2n+1)+2\sqrt{\frac{(4\alpha+1)^2}{4}+p''_0}\right]^2=(p''_1)^2 \tag{54}$$

And

$$p''_2=\frac{(E+Mc^2)}{\hbar^2c^2}[2q_2-(E-Mc^2)]$$

$$p''_1=-\frac{(E+Mc^2)}{\hbar^2c^2}[2v_0-4q_2] \tag{55}$$

$$p''_0=4\alpha^2k(k+1)$$

Substring p''_2, p''_1 and p''_0 in Eq. (54) we have

$$\frac{(E-Mc^2)}{\hbar^2c^2}[2q_2-(E+Mc^2)][(2n+1)+2\sqrt{\frac{(4\alpha+1)^2}{4}+4\alpha^2k(k+1)}]^2=\frac{(E+Mc^2)^2}{\hbar^4c^4}[2v_0-4q_2]^2 \tag{56}$$

After a few calculations we can obtain the energy Eigen-values equation, in form, as

$$E_{n,k}=\frac{2q_2\eta^2+(\eta^2-\gamma^2)Mc^2}{(\eta^2+\gamma^2)} \tag{57}$$

Where

$$\eta=\hbar c[(2n+1)+2\sqrt{\frac{(4\alpha+1)^2}{4}+4\alpha^2k(k+1)}] \tag{58}$$

$$\gamma=(2v_0-4q_2) \tag{59}$$

In Fig. 4, for the spin symmetry we show the behavior of the energy Eigen-values equation for three levels energy ($E_{1,0}$, $E_{2,1}$ and $E_{3,2}$) for some of screening parameter values (α) by taking the strength parameters q_1=q_2=v_0=v_1 0.8. It is seen that the energy Eigen-values increase with the increasing of the screening parameter value.

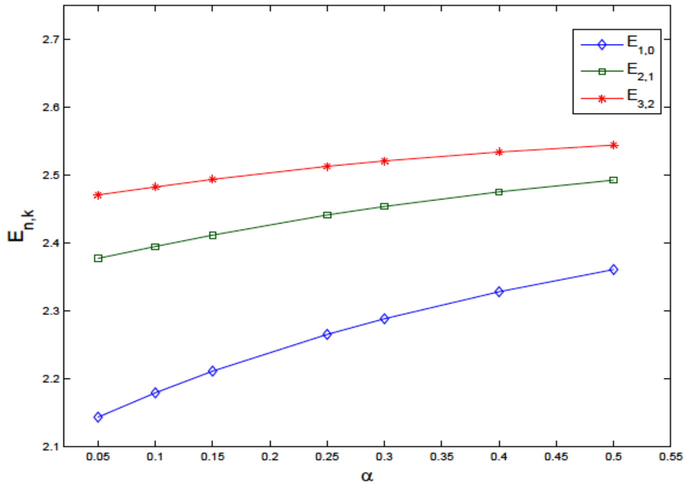

Fig. 4 The behavior of the energy Eigen-values for the spin symmetry

By using $F_{n_r,k}(r) = rR(r)$ and using the relation Eq. (13), we can finally obtain the wave functions with the aid of Equation 8 as

$$F_{n_r,k}(r) = Nr(1-e^{-2\alpha r})^{[\frac{(4\alpha+1)}{2}+\sqrt{p_0''+\frac{(4\alpha+1)^2}{4}}]} \exp[-\sqrt{p_2''}(1-e^{-2\alpha r})]L_n^{2\sqrt{p_0''+\frac{(4\alpha+1)^2}{4}}}[2\sqrt{p_2''}(1-e^{-2\alpha r})] \quad (60)$$

Where N is the normalization constant and, the lower component of the Dirac spinor can be calculated by Eq. (61) as

$$G_{n_r,k}(r) = \frac{\hbar^2 c^2}{E+Mc^2}\left(\frac{d}{dr}+\frac{k}{r}\right)F_{n_r,k}(r) \quad (61)$$

We have obtained the energy Eigen-values and the wave function of the radial Dirac equation for spatially modified Eckart plus Hulthen potential with the spin symmetry for k≠0.
Now we consider the condition of the pseudo spin symmetry, i.e., ∑(r) = 0 and we have obtained the energy Eigen-values and the wave function.
For pseudo spin symmetry we have:

$$\left(-\frac{d^2}{dr^2}+\frac{k(k-1)}{r^2}+\frac{1}{\hbar^2 c^2}[Mc^2-E][Mc^2+E-\Delta(r)]G_{n_r,k}(r)=0 \quad (62)$$

If we define a new variable $G_{n_r,k}(r) = rg(r)$ and substituting it in to Eq. (62), we obtain the radial equation of Dirac equation as

$$\left[\frac{d^2}{dr^2}+\frac{2}{r}+\frac{(E^2-M^2c^4)}{\hbar^2 c^2}-\frac{(E-Mc^2)}{\hbar^2 c^2}\Delta(r)-\frac{k(k-1)}{r^2}\right]g_{n_r,k}(r)=0 \quad (63)$$

By using the following pekeris-type approximation and using the transformation $s = (1-e(-2\alpha r))$ we can write the Eq. (63)as summarized below

$$g''(s)+\frac{(-4\alpha)}{s}g'(s)+\frac{1}{s^2}[-p_2'''s^2+p_1'''s-p_0''']g(s)=0 \quad (64)$$

And,

$$p_2''' = \frac{(E - Mc^2)}{\hbar^2 c^2}[2q_2 - (E + Mc^2)]$$
$$p_1''' = -\frac{(E - Mc^2)}{\hbar^2 c^2}[2v_0 - 4q_2] \qquad (65)$$
$$p_0''' = 4\alpha^2 k(k-1)$$

We can obtain the energy Eigen-values equation, in form as

$$E_{n,k} = \frac{2q_2\eta'^2 + (\gamma^2 - \eta'^2)Mc^2}{(\gamma^2 + \eta'^2)} \qquad (66)$$

Where

$$\eta' = \hbar c[(2n+1) + 2\sqrt{\frac{(4\alpha+1)^2}{4} + 4\alpha^2 k(k-1)}] \qquad (67)$$

$$\gamma = (2v_0 - 4q_2) \qquad (68)$$

In Fig. 5, For the pseudo spin symmetry we show the behavior of the energy Eigen-values equation for three levels energy ($E_{1,0}$, $E_{2,1}$ and $E_{3,2}$) for some of screening parameter values (α) by taking the strength parameters q_1=q_2=v_0=v_1 0.8. It is seen that the energy Eigen-values decreases with the increasing of the screening parameter value.

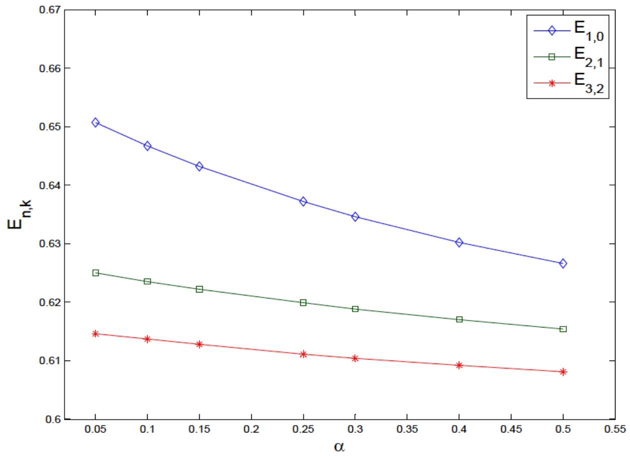

Fig. 5 The behavior of the energy Eigen-values for the pseudo spin symmetry

Using the Eq. (13), we can finally obtain the wave functions as

$$G_{n_r,k}(r) = Nr(1-e^{-2\alpha r})^{[\frac{(4\alpha+1)}{2} + \sqrt{p_0''' + \frac{(4\alpha+1)^2}{4}}]} \exp[-\sqrt{p_2'''}(1-e^{-2\alpha r})] L_n^{2\sqrt{p_0''' + \frac{(4\alpha+1)^2}{4}}}[2\sqrt{p_2'''}(1-e^{-2\alpha r})] \qquad (69)$$

Where N is the normalization constant, On the other hand, the lower component of the Dirac spinor can be calculated by Eq. (70) as

$$F_{n_r,k}(r) = \frac{\hbar^2 c^2}{Mc^2 - E}(\frac{d}{dr} - \frac{k}{r})G_{n_r,k}(r) \qquad (70)$$

We have obtained the energy Eigen-values and the wave function of the radial Dirac equation for spatially modified Eckart plus Hulthen potential with the pseudo spin symmetry for k≠0.

6 Conclusions

In this paper, we have discussed approximately the solutions of the Dirac equation for Eckart potential plus Hulthen potential with Spin Symmetry and Pseudo spin Symmetry for k≠0. We could obtain the energy Eigen-values and the wave function in terms of the generalized Laguerre polynomials functions via the PNU method. We have also considered the limiting cases of spin and pseudo spin symmetry for modified Eckart plus Hulthen potential to obtain the exact energy Eigen-values and the wave function. To show the accuracy of the present model, some numerical values of the energy levels are shown in figure 2, 3, 4 and 5. We can conclude that our results are interesting for experimental physicists, because the results are exact, more general and useful to study nuclear scattering, nuclear and particle physics.

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