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Spline collocation method for stochastic differential e uations

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Abstract: In this paper, we will present a new algorithm for solving Ito stochastic differential equations (SDEs) in continuous piecewise polynomial space. In this approach, we will employ piecewise collocation method for drift and diffusion terms of the given equation. Convergence order of the method is investigated and some numerical examples are considered to demonstrate the efficiency and robustness of the method.
Keywords: Stochastic differential equations; Spline collocation method; Piecewise polynomial space; Ito Integral; Euler-Maruyama method .

1. Introduction

Theory of stochastic differential equations as a relatively new field of science, is going to play a prominent role in the mathematical world. Indeed, an SDE is a differential equation in which one or more of the terms, and hence the solution itself, is a stochastic process. Ito in the late 50's has found wide range of applications including Biology, Chemistry, Mechanics, Economics and so on.

Based on the fact that most of SDE models don't posses an explicit exact analytic solution, it is necessary to drive numerical methods to generate an approximate solution to the problem under consideration.

One of the key methods that employ to understand SDEs is Monte-Carlo simulation. This method involves generating many sample paths or sequences of random variable that are distributed in some known way. We consider models based upon the standard Brownian motion. Based on this strategy, we will construct an efficient numerical method to obtain a numerical solution of the given SDE, in direct analogy with the deterministic methods such as Euler method.

We organize this paper as follows: In section 2, we will define the Brownian motion (Wiener process) and will express two main stochastic integrals (Ito and Stratonovich integrals) and then Euler-Maruyama method for finding the numerical solution of SDEs is introduced. In the sequel, existence and uniqueness of the solution and convergence properties of the method is discussed. In section 3, we will refer to piecewise polynomial collocation method for integrating ordinary differential equations (ODEs). After construction a new numerical method in section 4, we will present some computational experiments in next section.

2. Stochastic Differential Equations

In this section the fundamental concepts of Euler-Maruyama method and its convergence for SDEs is discussed.

A. *Brownian Motion*

A scalar Brownian motion, or standard Wiener process, over $[0,T]$ is a random variable $w(t)$ that depends continuously upon $t \in [0,T]$ and satisfies:

1) $w(0) = 0$, with probability 1.
2) For $0 \le s < t \le T$ the random variable given by increment $w(t) - w(s)$ is normally distributed with mean zero and variance $t - s$ i.e.
   $w(t) - w(s) \sim \sqrt{t-s}\, N(0,1)$,

where $N(0,1)$ denotes a normally distributed random variable with zero mean and unit variance.

3) For $0 \le s < t < u < v \le T$, $w(t) - w(s)$ and $w(v) - w(u)$ are independent.

Fig.1 shows a Wiener process for n=1000.

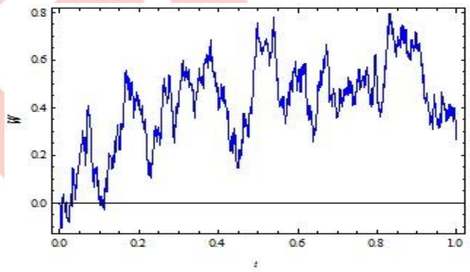

Figure 1: Brownian path Sample.

B. *Stochastic Integrals*

A stochastic integral is the integral of some function $h(t)$ over some interval $[0,T]$, but with respect to a Brownian motion $w(t)$ as

$$\int_0^T h(t)\,dw_t . \qquad (1)$$

We approximate (1) as ordinary integral by the Riemann sum:

$$\int_0^T h(t)\,dw_t \approx \sum_{j=0}^{N-1} h(t_j^*)(w(t_{j+1}) - w(t_j)) .$$

In the deterministic setting, answer of (1) isn't

different if we choose t_j^* from Trapezoidal or midpoint rules, but in the stochastic form, we will two rules with two different answers:

1) Ito Integral: With $t_j^* = t_j$, we have:

$$\int_0^T h(t)dw_t \approx \sum_{j=0}^{N-1} h(t_j)(w(t_{j+1}) - w(t_j)),$$

2) Stratonovich Integral: When we choose

$$t_j^* = \frac{t_j + t_{j+1}}{2},$$

we have

$$\int_0^T h(t) \circ dw_t \approx \sum_{j=0}^{N-1} h(\frac{t_j + t_{j+1}}{2})(w(t_{j+1}) - w(t_j)).$$

The choice of which interpretation (Ito or Stratonovich) should be used, depends on the type of analysis required for an SDE and in this paper the Ito form will be used. In order to avoid any confusion in notation, henceforth the symbol $\circ$ will be used to denote the Stratonovich form. For another properties of stochastic integrals see [4].

C. The Euler-Maruyama Method

The general form of the SDE which will be considered in this paper, is given by

$$dx_t = f(t, x_t)dt + g(t, x_t)dw_t, \qquad (2)$$

$$x(t_0) = x_0,\ x \in \mathbb{R}^m.$$

Here f is an m-vector-valued function, g is an $m \times p$ matrix-valued function and $w(t)$ is a p-dimensional process having independent scalar Wiener process components ($t \geq 0$) and the solution $x(t)$ is an m-vector process. In SDE (2) f is called drift coefficient and g is called diffusion coefficient (or noise term).

A solution to (2) is a process $X_t(w)$ that is an adapted function of W, so that

$$x(t) = x(t_0) + \int_{t_0}^t f(s, x_s)ds + \int_{t_0}^t g(s, x_s)dw_s. \qquad (3)$$

In the integral formulation, $w(t)$ is a Wiener process and can be interpreted in such a way that the derivative of W is the Gaussian white noise process (in fact $w(t)$ is not differentiable).

To apply a numerical scheme to the SDE (3), we must firstly discretize time interval $[0, T]$ by using a fixed step-size $h = \frac{T}{N}$. This gives us a set of equally spaced points as:

$$0 = t_0 < t_1 < \ldots < t_n < \ldots < t_N = T,$$

for approximation our numerical solution.

The Euler-Maruyama (E-M) method is so far the most studied numerical method for solving SDEs which takes the form (in one dimension):

$$x_{n+1} = x_n + h_n f(x_n) + \Delta w_n g(x_n),\ n = 0, 1, \ldots, N-1$$

(4)

Where $h_n = t_{n+1} - t_n$ and $\Delta w_n = w(t_{n+1}) - w(t_n)$. From definition of a wiener process it follows that these increments are independent with normal distribution $N(0, h_n)$.

In examining the first three terms of the stochastic Taylor expansion, we see that this form is the basis of Euler-Maruyama scheme. When $g(x) = 0$, this reduces to the ordinary deterministic Euler scheme. We can adding more and more stochastic terms from the stochastic Taylor expansion and obtain more accurate methods.

Fig. 2 shows the exact solution (with red stars) and Euler-Maruyama method (with blue mesh).

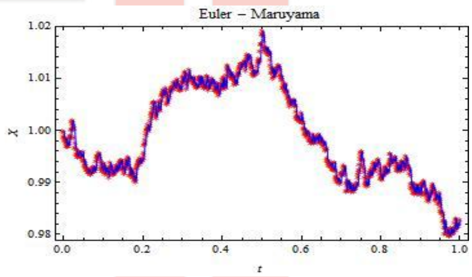

Figure 2: Euler-Maruyama approximation.

We now express existence and uniqueness theorem and in the sequel, refer to the convergence definitions.

Theorem 1: Let $T > 0$ and $f(.,.) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, $g(.,.) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ be measurable functions satisfying:

$$|f(t, x)| + |g(t, x)| \leq C(1 + |x|), \qquad x \in \mathbb{R}^n, t \in [0, T]$$

for some constant C and

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq D|x - y|,$$

$x, y \in \mathbb{R}^n$,

for some constant D. Let $x_0 = Z$ be a random variable which is independent of the σ-algebra $F_\infty^{(m)}$ generated by $w_s(.)$, $s \geq 0$ such that $E(|Z|^2) < \infty$.

Then the SDE (2) has a unique t-continuous solution x_t with the property that x_t is adapted to the filtration F_t^z generated by Z and $w_s(.)\,; s \leq t$ and

$$E[\int_0^T |x_t|^2 dt] < \infty.$$

Proof: see [4,6,7].

Some numerical time-discretization methods for the numerical solution of SDEs, briefly have already been

discussed in this paper. In order to evaluate the efficacy of such methods, two ways of measuring accuracy are used: strong convergence and weak convergence.

Definition 1: We say that a discrete time approximation Y^h converges strongly with order $p>0$ at time T if there exists a positive constant C, which does not depends on the maximum step size h and $\delta_0>0$, such that:

$$E(\|X_T - Y_T^h\|) \le Ch^p,$$

for each $h=\frac{T-t_0}{N}\in(0,\delta_0)$, N is the number of subintervals in $I=[t_0,T]$, X_T and Y_T are the exact and the approximate solution at T, respectively.

It can be seen from literature, the Euler-Maruyama scheme has strong order of convergence p=0.5.

In some cases, it is not necessary to find an accurate path wise approximation of an Ito process. Instead, only some of the moments may be of interest or, more generally, $E(f(X))$ for some function $f(X)$. This is a much weaker condition.

Definition 2: A discrete time approximation Y_T with maximum step size h is said to be converges weakly with order $p>0$ to X at time T as $h\to 0$, if for each $f\in C_p^{2(p+1)}(R^d,R)$ there exists positive constant C, which does not depends on h and a finite number $\delta_0>0$, such that

$$|E(f(X_T)) - E(f(Y_T^h))| \le Ch^p,$$

for each $h\in(0,\delta_0)$ ([6]).

3. Piecewise Collocation Method

Consider the initial-value problem:

$$x'(t) = f(t, x(t)), \qquad t\in I := [0,T],$$
$$x(0) = x_0, \tag{5}$$

and assume that the Lipschitz-continuous function $f: I\times\Omega\subset\mathbb{R}\to\mathbb{R}$ is such that (4) possesses a unique solution $y\in C^1(I)$ for all $x_0\in\Omega$. Let

$$I_h := \{t_n : 0=t_0<t_1<\dots<t_N=T\},$$

be a given (not necessarily uniform) mesh on I and set $\sigma_n := (t_n, t_{n+1}]$, $\bar{\sigma}_n := [t_n, t_{n+1}]$, with $h_n = t_{n+1}-t_n$ $(n=0,1,\dots N-1)$, $h = Max\{h_n : 0\le n\le N-1\}$.

Suppose:

$$x(t) = x_0 + \int_0^t f(s, x_s)ds, \qquad t\in I. \tag{6}$$

We use the Volterra integral equation (6) as the basis for obtaining collocation approximation to the solution x of (5). Denote by

$$S_{m-1}^{(-1)}(I_h) := \{\upsilon : \upsilon|_{\sigma_n} \in \pi_{m-1}(0\le n\le N-1)\},$$

the space of piecewise polynomials of degree $m-1\ge 0$ which may be discontinuous at the interior points $t_1, t_2, \dots, t_{N-1}$ of the mesh I_h. More precisely, let Y_h be given by

$$Y_h = \{t = t_n + c_i h : 0\le c_1<\dots<1 (0\le n\le N-1)\}, \tag{7}$$

for a given mesh I_h. So, the collocation parameters $\{c_i\}$ completely determine Y_h. The collocation solution $\upsilon_h\in S_{m-1}^{(-1)}(I_h)$ for (6) is given locally by

$$\upsilon_h(t_n+\upsilon h) = \sum_{j=1}^{m} L_j(\upsilon)V_{n,j}, \qquad \upsilon\in(0,1] \tag{8}$$

with $V_{n,j} := \upsilon_h(t_n + c_j h)$ and is defined by the ollocation equation:

$$\upsilon_h(t) = X_0 + \int_0^t f(s,\upsilon_h(s))ds, \quad t\in X_h.$$

Setting

$$F_n := \int_0^t f(s,\upsilon_h(s))ds = \sum_{l=0}^{n-1} h_l\int_0^1 f(t_l+sh_l, \upsilon_h(t_l+sh_l))ds,$$

and $t = t_{n,i} := t_n + c_i h_n$,

(8) may be written in the form

$$V_{n,i} = X_0 + F_n + h_n\int_0^{c_i} f(t_n+sh_n, \upsilon_h(t_n+sh_n))ds$$
$$= X_0 + F_n + h_n\int_0^{c_i} f(t_n+sh_n, \sum_{j=1}^{m} L_j(s)V_{n,j})ds. \tag{9}$$

For more details see [1].

Now we explain the above spline collocation method for SDEs:

Let us consider the given SDE as

$$dx(t) = f(x(t))dt + g(x(t))dw(t), \quad t\in[0,T]. \tag{10}$$

Consider a set of equally spaced points

$$0 = t_0 < t_1 < \dots < t_N = T.$$

We rewrite (10) as

$$x(t) = x(t_0) + \int_0^T f(x(s))ds + \int_0^T g(x(s))dw(s). \tag{11}$$

Inserting the collocation parameters $\{c_i\}$, $i=0,1,\dots m$ into (11) yields

$$x(t_n+c_jh) = x(t_0) + \int_0^{t_n+c_jh} f(x(s))ds + \int_0^{t_n+c_jh} g(x(s))dw(s).$$

Now, dividing the interval $[0, t_n+c_jh]$ to $[0,t_1]\cup[t_1,t_2]\cup\dots\cup[t_n, t_n+c_jh]$ and using collocation scheme (9), gives us a numerical approximation for the first integral term. For the second integral, we use the Ito integral and make a Wiener process with mN knots as

$$\int_0^T g(x(s))dw(s) \approx \sum_{j=0}^{N-1}\int_{t_j}^{t_{j+1}} g(x(s))dw(s) + \int_{t_n}^{t_n+c_jh} g(x(s))dw(s) \tag{12}$$

$$= \sum_{i=0}^{N-1}(g(x(t_i))(w(t_{i+1})-w(t_i))) + g(x(t_n))(w(t_n+c_jh)-w(t_n)).$$

For determining the order of convergence, as pointed by many authors in literature, we consider proposed collocation method for 100 Brownian path with different step-sizes in the end point. We prove that its strong convergence rate is of the same order as the Euler-Maruyama method. For more details, see [8].

4. Numerical Results

To see how well the Spline collocation scheme (9,12) works, we compare its performance with the standard E-M scheme(4) by solving two sample stochastic differential equations. The numerical result of these methods will be shown in Table 1 and 2.

Example 1: The first example that we are going to test is the following linear SDEs:

$dx(t) = 0.01x\,dt + 0.02x\,dw(t)$, $x_0 = 1$,

where the exact solution is given by

$$x(t) = x_0 e^{\left(0.01 - \frac{(0.02)^2}{2}t + 0.02w(t)\right)}.$$

The numerical solution using the Spline collocation method for different m (collocation points) which is discussed in this paper and E-M method with the maximum error are given in Table 1.

In order to clearly demonstrate the convergence rate of the Spline collocation method, we also plotted the average sample errors at the terminal time T for the Spline collocation scheme with different step-sizes in Fig 3.

TABLE 1
MAX. ERROR OF THE EULER-MARUYAMA METHOD AND THE SPLINE COLLOCATION METHOD FOR EXAMPLE 1.

N	m=4	m=5	Euler-Maruyama
2^5	2.52E-5	1.83E-5	2.86E-5
2^6	2.94E-5	1.51E-5	5.94E-5
2^7	2.14E-5	1.80E-5	1.61E-5
2^8	1.84E-5	1.08E-5	2.76E-5
2^9	5.68E-6	9.23E-6	2.36E-5

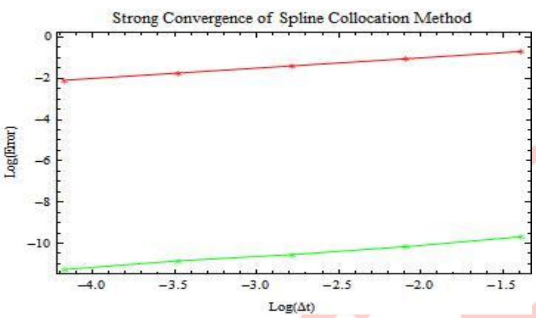

Figure 3: The convergence rate of the Spline collocation method.

Example 2: Consider nonlinear SDE

$$dx(t) = -\frac{1}{2}q^2 x\,dt + q\sqrt{1 - x^2}\,dw(t) ,$$

with q=0.02 and $x_0 = 0$, where the exact solution is

$$x(t) = \sin(qw(t) + \sin^{-1}(x_0)) .$$

The obtained numerical results by using the Spline collocation method for different collocation parameters and E-M method for nonlinear equation 2 have reported in Table 2.

From the figure 3 and tables, it is easy to see that both the E-M and Spline collocation schemes have the half order convergence, but the Spline collocation scheme obtains better approximate solutions in comparison to E-M method.

Table 2
MAX. ERROR OF THE EULER-MARUYAMA METHOD AND THE SPLINE COLLOCATION METHOD FOR THE NONLINEAR EXAMPLE 2.

N	m=4	m=5	Euler-Maruyama
2^5	1.916E-7	2.749E-7	3.676E-7
2^6	2.554E-7	1.024E-7	3.197E-7
2^7	7.512E-8	9.584E-8	1.448E-7
2^8	1.439E-7	1.279E-7	3.766E-7
2^9	2.627E-7	8.843E-8	5.296E-8

5. Conclusion

In this paper, we proposed and discussed a Spline collocation method for numerical solution of the stochastic differential equations driven by the one-dimensional Brownian motion. Although the strong convergence rate of our method is of the same order as that of the Euler-Maruyama method, this scheme makes it possible to obtain better approximation than standard approach.

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