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Fuzzy Random Inventory Model with Possibility and Probability Constraints

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Abstract

In this paper, one of the models of inventory control problem is introduced. The model has warehouse and budgetary constraints. In warehouse constraint, the parameters are fuzzy triangular number while, in budgetary constraint, available total budgetary capital is fuzzy random variable. Considering the possibility levels, constraints are defuzzified. The budgetary constraint is converted to crisp one which is neither fuzzy nor stochastic by the chance-constrained programming approach. Finally, we have an optimal control problem with two crisp constraints which is solved by the Pontryagin Minimum Principle and Khun-Taker conditions. The model has been illustrated by numerical example.

Keywords: Fuzzy random variable, Chance-constrained programming, Possibility level

1. Introduction

Many researchers have developed the decision making problems in fuzzy or stochastic environments. Charns and Cooper [2] introduced the chance-constrained programming technique to solve a linear programming problem with the probability constraints. Also, Dubois and Prade [3] introduced the possibility constraints for defuzzification process of fuzzy decision making problems. Liu [5, 6] and chakraborty [1] have introduced some models of decision making where possibility levels and probability measures are assigned to the constraints. Besides, he has applied the fuzzy simulation to solve them. During the last few years, some inventory models [7, 8] have been formulated in the fuzzy and stochastic environment. Researchers have considered fuzzy and/or stochastic parameters in a single model. In these models, constraints were not composed in possibility and probability senses at the same time. In some practical systems, the outcomes of random variable may not be numerical ones, but they can be vague in linguistic terms. The fuzzy random variable describes the uncertain parameters in which fuzziness and randomness are fused with each other. For instance, in inventory control systems, we may encounter uncertain and random budgetary capital during the period of business so that we formulate it by a fuzzy random variable. In these cases, we cannot formulate the parameter as a fuzzy number or a random variable. So, I have proposed an inventory control model in which, available total budgetary capital is a fuzzy random variable. Meanwhile, storage area per unit item and available maximum space of storage are fuzzy triangular numbers. The constraints are defuzzified based on Dubois and Prade [3] as done by Liu [5, 6]. The outcome stochastic budgetary constraint is converted to the optimal control problem having crisp constraints by chance-constrained programming approach [2]. The problem is solved using Pontryagin Minimum Principle and Kuhn-Taker conditions.

In section 2, some necessary definitions and concepts are stated. Section 3 introduces the model and deals with the process of converting it to crisp model. Section 4 presents the mathematical approach to solve the model. To illustrate the process, a numerical example is given in section 5. Section 6 closes with conclusion.

2. Preliminaries

At first, we shall state some necessary concepts, definitions and lemmas on fuzzy environment.

Definition 2.1 Triangular fuzzy number $\tilde{a}$ is the fuzzy number with the membership function $\mu_{\tilde{a}}(x)$, a continuous mapping $\mu_{\tilde{a}}(x): R \to [0,1]$

$$\mu_{\tilde{a}}(x) = \begin{cases} 0 & , \quad x \le a^l \\ \dfrac{x-a^l}{a^m-a^l} & , \ a^l \le x \le a^m \\ \dfrac{a^u-x}{a^u-a^m} & , \ a^m \le x \le a^u \\ 0 & , \ a^u \le x \end{cases}$$

The β-level set of $\tilde{a}$ is the set

$$\tilde{a}_\beta = \left\{ x \in R \,\middle|\, \tilde{a}(x) \ge \beta \right\},$$

where $\beta \in (0,1]$.

We denote the set of all fuzzy numbers by $F_\circ(R)$.

Definition 2.2 ([3]) Let $\tilde{a}$ and $\tilde{b}$ be fuzzy numbers with membership functions $\mu_{\tilde{a}}(x)$ and $\mu_{\tilde{b}}(x)$, respectively. Then

$$Pos(\tilde{a} \le \tilde{b}) = \sup_{x \le y} \min(\mu_{\tilde{a}}(x), \mu_{\tilde{b}}(y))$$

where Pos represents the possibility.

Lemma 2.1 ([3]) Let $\tilde{a} = (a^l, a^m, a^u)$ and $\tilde{b} = (b^l, b^m, b^u)$ be triangular fuzzy numbers. Then

$$Pos(\tilde{a} \le \tilde{b}) \ge \beta \quad \text{iff} \quad \frac{b^u - a^l}{a^m - a^l + b^u - b^m} \ge \beta, \quad (a^m > b^m, b^u > a^l)$$

Definition 2.3 ([4]) Let (Ω, A, P) be a probability measure space. The mapping $\tilde{a}: \Omega \to F_\circ(R)$ is called a fuzzy random variable on (Ω, A, P), if for any $\alpha \in (0,1]$ and $\omega \in \Omega$,

$$\tilde{a}_\alpha(\omega) = \left\{ x \,\middle|\, x \in R \ , \ \tilde{a}(\omega)(x) \ge \alpha \right\} = \left[a_\alpha^-(\omega), a_\alpha^+(\omega) \right]$$

be a random interval, namely, $a_\alpha^-(\omega)$ and $a_\alpha^+(\omega)$ are random variables (or finite measurable functions) on (Ω, A, P).

Definition 2.4. Consider the following problem:

$$Min\ J=\int_0^T\left(\sum_{j=1}^{n}h_jx_j(t)+\sum_{r=1}^{p}c_ru_r(t)\right)dt \tag{1}$$

s.t.

$$\dot{x}_i(t)=\sum_{j=1}^{n}a_{ij}x_j(t)+\sum_{r=1}^{p}b_{ir}u_r(t) \quad ,i=1,\dots,n \tag{2}$$

$$\sum_{j=1}^{n}h_{kj}\ x_j(t)+\sum_{r=1}^{p}c_{kr}u_r(t)\le M_k \quad ;k=1,\dots,l \tag{3}$$

$$x_i(t_0)=x_{i0}\ ,0\le u_r^l\le u_r(t)\le u_r^u\ ,0\le t\le T \tag{4}$$

where $x_i(t)$, i=1,…,n and $u_r(t)$, r=1,…,p are state and control variables respectively. $a_{ij}, b_{ir}, h_{kj}, c_{kr}$ and M_k , $i=1,\dots,n; k=1,\dots,l$ are coefficients of stat and control variables and T is length of the finite time.

The problem is to find the optimal control $u(t)$ and the corresponding trajectory $x(t)$, $0\le t\le T$, which satisfy control equalities (2) and constraints (3) while minimizing the performance index J. This problem is named the optimal control with constraints.

3. Optimal control problem for inventory model in uncertainty environment

The proposed model is similar to the Maity and Maiti model in [8] with some variations. A defective n-item production-inventory system is considered. The system has budget and warehouse capacity constraints. The items are produced at a variable rate $u_r(t)$ in which δ_i fraction is defective. The demand of the items is time independent and the stock level at time t decreases due to defectiveness and consumption. Shortages are allowed but may or may not be backlogged. Maximum space available of storage is a fuzzy triangular number. Meanwhile, Available total budgetary capital is a fuzzy random variable related to ω which has normal distribution with known mean and variance. The space and budgetary constraints are satisfied with some predefined possibility and probability leveles, respectively. Minimization of total costs consists of holding, shortage and production costs leads to the following optimal control problem with constraints:

$$Min\ J=\sum_{j=1}^{n}\int_0^T\left(h_jI_j(t)+h'_jS_j(t)+c_ju_j(t)\right)dt \tag{5}$$

s.t.

$$\dot{x}_j(t)=(1-\delta_j)u_j(t)-d_j \quad ,j=1\dots,n \tag{6}$$

$$\sum_{j=1}^{n}\tilde{a}_jI_j(t)\le\tilde{M} \tag{7}$$

$$\sum_{j=1}^{n}c_ju_j(t)\le\tilde{Z}(\omega) \tag{8}$$

$$x_j(0)=0\ ,0\le d_j\le u_j(t)\le u_j\ ,0\le t\le T \tag{9}$$

The equation (6) is the differential equation for ith item during a fixed time-horizon, T. (7) and (8) are warehouse and budgetary constraints respectively. The model notations are:

N: Number of items

$\widetilde{M}$: Maximum space available of storage

$\widetilde{Z}(\omega)$: Available total budgetary capital that is fuzzy random variable related to ω .

$u_j(t)$: Production rate at time t, u_j : Maximum production rate

$x_j(t)$: Inventory level at time t, $I_j(t) = \max(x_j(t), 0)$,

$S_j(t) = \max(-x_j(t), 0)$: shortage level at time t. We have $x_j(t) = I_j(t) - S_j(t)$

δ_j : Fraction defective

$\widetilde{a}_j$: Storage area per unit item that is fuzzy triangular number

d_j : Demand

c_j : Production cost per unit item, h_j : Holding cost per unit item, h'_j : Shortage cost per unit item

T: Length of the finite time horizon.

There are a representation of stochastic constraints and several representations of fuzzy constraints. In literature, a level of probability is assigned to the stochastic constraints and is used the chance-constrained programming approach for converting them to the crisp constraints. In the some models, the fuzzy constraints are represented in the setting of possibility theory in which fuzzy numbers are interpreted by the degree of uncertainty [7 and 8]. The imprecise warehouse constraint (7) is written under possibility constraint as

$$Pos\left(\sum_{j=1}^{n} \widetilde{a}_j I_j(t) \le \widetilde{M}\right) \ge \beta_1 \tag{10}$$

where β_1 is the possibility level.

But in proposed model, (8) is a fuzzy stochastic constraint because the budgetary capital is a fuzzy random variable related to ω . According to Liu [5, 6], for the imprecise budgetary constraint (8), we can consider the possibility and probability levels at the same time. Then (8) is written as

$$\Pr\left(Pos\left(\sum_{j=1}^{n} c_j u_j(t) \le \widetilde{Z}(\omega)\right) \ge \beta_2 \right) \ge \alpha \tag{11}$$

where α and β_2 are predetermined probability and possibility levels.

Let $\widetilde{a}_j = (a_j^l, a_j^m, a_j^u)$, $\widetilde{M} = (M^l, M^m, M^u)$ and $\widetilde{Z}(\omega) = (Z^l(\omega), Z^m(\omega), Z^u(\omega))$ be triangular fuzzy numbers. Then using the lemma 2.1, the possibility explanations in (10) and (11) convert to the following inequalities:

$$\frac{M^u - \sum_{j=1}^{n} a_j^l I_j(t)}{\sum_{j=1}^{n} a_j^m I_j(t) - \sum_{j=1}^{n} a_j^l I_j(t) + M^u - M^m} \ge \beta_1 \tag{12}$$

$$\frac{Z^u(\omega) - \sum_{j=1}^{p} c_j u_j(t)}{Z^u(\omega) - Z^m(\omega)} \ge \beta_2 \tag{13}$$

Now, we can rewrite (12) and (13) as follow:

$$(1-\beta_1)\sum_{j=1}^{n} a_j^l\, I_j(t) + \beta_1 \sum_{j=1}^{n} a_j^m\, I_j(t) \le (1-\beta_1)M^u + \beta_1 M^m \quad (14)$$

$$\Pr\left(\sum_{j=1}^{p} c_j u_j(t) \le (1-\beta_2)Z^u(\omega) + \beta_2 Z^m(\omega)\right) \ge \alpha \quad (15)$$

Constraint (14) is a crisp one while (15) is stochastic. Optimal control problem with constraint (15) is a probability programming problem. Hence, it can be converted to the crisp (neither fuzzy nor stochastic) model by the classic technique of chance-constrained programming [2]. Therefore, we have the following optimal control problem:

$$Min\ \ J = \sum_{j=1}^{n} \int_0^T (h_j I_j(t) + h_i' S_i(t) + c_j u_j(t))dt$$

s.t.

$$\dot{x}_j(t) = (1-\delta_j)u_j(t) - d_j \quad , j = 1\ldots,n$$

$$\sum_{j=1}^{n}((1-\beta_1)a_j^l + \beta_1 a_j^m)\, I_j(t) \le (1-\beta_1)M^u + \beta_1 M^m \quad (16)$$

$$\sum_{j=1}^{p} c_j u_j(t) - \varphi^{-1}(1\text{-}\alpha)\sqrt{\mathrm{var}((1-\beta_2)Z^u(\omega) + \beta_2 Z^m(\omega))} \le (1-\beta_2)Z^u(\varpi) + \beta_2 Z^m(\varpi) \quad (17)$$

$$x_j(0) = 0\,, 0 \le d_j \le u_j(t) \le u_j\,, 0 \le t \le T\,, \alpha \in (0,1]$$

where ϖ is the mean of ω , φ is the standard normal distribution function, φ^{-1} is inverse of φ and *var* denote the variance.

4. The solving Process

The corresponding Hamiltonian function of model is

$$\mathrm{H} = \sum_{j=1}^{n}(h_j I_j(t) + h_j' S_j(t) + c_j u_j(t)) + p_j(t)((1-\delta_j)u_j(t) - d_j)$$

The Lagrangian function for the constraint (17) is

$$L = H + \lambda_1\left(\sum_{j=1}^{n}((1-\beta_1)a_j^l + \beta_1 a_j^m)\, I_j(t) - (1-\beta_1)M^u - \beta_1 M^m\right)$$
$$+ \lambda_2\left(\sum_{j=1}^{p} c_j u_j(t) - \varphi^{-1}(1\text{-}\alpha)\sqrt{\mathrm{var}((1-\beta_2)Z^u(\omega) + \beta_2 Z^m(\omega))} - (1-\beta_2)Z^u(\varpi) - \beta_2 Z^m(\varpi)\right)$$

where $\lambda_1, \lambda_2 \ge 0$ are the Lagrange multipliers.

Hence, the Kuhn-Tucker conditions are

$$\lambda_1\left(\sum_{j=1}^{n}((1-\beta_1)a_j^l + \beta_1 a_j^m)\, I_j(t) - (1-\beta_1)M^u - \beta_1 M^m\right) = 0$$

$$\lambda_2\left(\sum_{j=1}^{p} c_j u_j(t) - \varphi^{-1}(1\text{-}\alpha)\sqrt{\mathrm{var}((1-\beta_2)Z^u(\omega) + \beta_2 Z^m(\omega))} - (1-\beta_2)Z^u(\varpi) - \beta_2 Z^m(\varpi)\right) = 0$$

The corresponding costate $p_j(t)$ is given by first order differential equation,

$$\dot{p}_j(t) = -\frac{\partial L}{\partial x_j}$$

$$\dot{p}_i(t) = \begin{cases} -h_j - \lambda_1((1-\beta_1)a_j^l - \beta_1 a_j^m) & x_j \geq 0 \\ -h'_j & x_j < 0 \\ k_j,\ k_j \in (-h'_j, -h_j) & x_j = 0 \end{cases} \tag{18}$$

$$p_i(T) = 0$$

By Pontryagin Minimum Principle [9], we have

$$\frac{\partial L}{\partial u_j(t)} = (1+\lambda_2)c_j + p_j(t)(1-\delta_j) \tag{19}$$

$u_j(t)$ is continuous and bounded. Therefore by (12) there are three cases:

$$\begin{cases} \frac{\partial L}{\partial u_j(t)} > 0 & L \text{ is increasing} \\ \frac{\partial L}{\partial u_j(t)} = 0 & L \text{ is independent} \\ \frac{\partial L}{\partial u_j(t)} < 0 & L \text{ is decreasing} \end{cases}$$

So, for $0 \leq d_j \leq u_j(t) \leq u_j$ we have:

$$u_j(t) = \begin{cases} u_j & \text{if } \ p_j(t) > \frac{-(1-\lambda_2)c_j}{(1-\delta_j)} & (20) \\ d_j & \text{if } \ p_j(t) < \frac{-(1-\lambda_2)c_j}{(1-\delta_j)} & (21) \end{cases}$$

Suppose that we have (20) for $0 \leq t \leq t_{i1}$ and (21) for $t_{i1} \leq t \leq T$, then from (6) we have

$$\dot{x}_j(t) = (1-\delta_j)u_j - d_j, \quad 0 \leq t \leq t_{i1} \tag{22}$$

$$\dot{x}_j(t) = -\delta_j d_j, \quad t_{i1} \leq t \leq T \tag{23}$$

Let initially for $0 \leq t \leq t_{i1}$, the stock is increasing and for $t_{i1} \leq t \leq T$ it decreases such that at t_{i2} the shortages allowed. Using (22) and (23), we find the optimum stock and shortage functions as

$$I_j(t) = I_j(0) + ((1-\delta_j)u_j - d_j)t, \quad 0 \leq t \leq t_{i1} \tag{24}$$

$$I_j(t) = I_j(t_{i1}) - \delta_j d_j(t - t_{i1}), \quad t_{i1} \leq t \leq T \tag{25}$$

$$S_j(t) = \delta_j d_j(t - t_{i1}), \quad t_{i2} \leq t \leq T \tag{26}$$

Therefore, we can obtain the minimum value of J as follows:

$$J = h_j \int_0^{t_{i1}} (I_j(0) + ((1-\delta_j)u_j - d_j)t)dt + h_j \int_{t_{i1}}^{t_{i2}} (I_j(t_{i1}) - \delta_j d_j (t - t_{i1}))\mathrm{d}t + h'_j \int_{t_{i2}}^{T} \delta_j d_j (t - t_{i1})\mathrm{d}t$$

$$+ c_j \left(\int_0^{t_{i1}} u_j dt + \int_{t_{i1}}^{T} d_j dt \right)$$

$$J = \sum_{j=1}^{n} h_j ((1-\delta_j)u_j - d_j)t_{i1} + \sum_{j=1}^{n} h_j \left(I_j(t_{i1}) - \frac{1}{2}\delta_j d_j (t_{i2} + t_{i1}) - t_{i1} \right)(t_{i2} - t_{i1})$$
$$+ \sum_{j=1}^{n} h'_j \delta_j d_j \left(\frac{1}{2}(T^2 - t_{i2}^2) - t_{i1}(T - t_{i2}) \right) + \sum_{j=1}^{n} c_j (u_j t_{i1} + d_j (T - t_{i1})) \qquad (27)$$

5. Illustrative example

To illustrate the proposed inventory model, an inventory system of two items with the numeric parameters is given. Suppose that, available maximum space of storage is fuzzy triangular number $\widetilde{M} = (85, 100, 115)$, available total budgetary capital is fuzzy random variable $\widetilde{Z}(\omega) = (\omega - 25, \omega, \omega + 25)$ where ω has normal distribution with mean 500 and variance 100, T=10, probability measure is $\alpha = 0.95$, possibility levels are $\beta_1 = 0.65$ and $\beta_2 = 0.67$, storage area for items are fuzzy triangular numbers $\widetilde{a}_1 = (1.9, 2.0, 2.1)$ and $\widetilde{a}_2 = (2.4, 2.5, 2.6)$, defective fractions $\delta_1 = 0.18$, $\delta_2 = 0.19$, maximum production rates $u_1 = 12$, $u_2 = 13$, demands $d_1 = 8.5$, $d_2 = 9$, production costs $c_1 = 2$, $c_2 = 2.5$, holding costs $h_1 = 0.5$, $h_2 = 0.6$ and shortage costs $h'_1 = 0.6$, $h'_2 = 0.5$.

Solving process:

According to (14) and (15), defuzzified constraints are:

$$20.065 I_1(t) + 2.565 I_2(t) \le 90.25 \qquad (28)$$
$$Pr\left(2u_1(t) + 2.5u_2(t) \le \omega - 16.75\right) \ge 0.95 \qquad (29)$$

The constraint (29) is converted to the following crisp constraint by (17):

$$2u_1(t) + 2.5u_2(t) - 506.55 \le 0 \qquad (30)$$

Now, we have the following problem:

$$Min\ J = \int_0^{10} \left(0.5 I_1(t) + 0.6 I_2(t) + 0.6 S_1(t) + 0.5 S_2(t) + 2u_1(t) + 2.5u_2(t)\right)dt$$

s.t.

$$\dot{x}_1 = 0.82u_1 - 8.5$$
$$\dot{x}_2 = 0.81u_2 - 9$$
$$20.065 I_1(t) + 2.565 I_2(t) \le 90.25$$
$$2u_1(t) + 2.5u_2(t) - 506.55 \le 0$$
$$x_1(0) = x_1(10) = 0,\ x_2(0) = x_2(10) = 0,$$
$$8.5 \le u_1 \le 12, 9 \le u_2 \le 13,$$
$$0 \le t \le 10$$

The optimal control function is obtained as,

$$u_1(t)=\begin{cases}12 & ,0\le t\le 5.53\\ 8.5 & ,5.53\le t\le 10\end{cases} \quad \& \quad u_2(t)=\begin{cases}13 & ,0\le t\le 5.28\\ 9 & ,5.28\le t\le 10\end{cases}$$

and we have,

$$I_1(t)=\begin{cases}1.34t & ,\ 0\le t\le 5.33\\ -1.53(t-5.33)+7.14 & ,\ 5.33\le t\le 10\end{cases} \quad \& \quad I_2(t)=\begin{cases}1.53t & ,\ 0\le t\le 5.28\\ -1.71(t-5.28)+8.08 & ,\ 5.28\le t\le 10\end{cases}$$

$$S_1(t)=S_2(t)=0$$

Then the corresponding trajectory is

$$x_j(t)=I_j(t)-S_j(t)=I_j(t)$$

So, we obtain $J^*=473.076$.

6. Conclusion

In this paper, a model of inventory control problem with fuzzy stochastic constraints was introduced and dealt with. The possibility levels for storage and budgetary constraints and probability measure for budgetary one were assigned. Possibility theory and chance-constrained programming approach was employed for converting the imprecise model to the crisp one. To solve the generated optimal control problem, Pontryagin Minimum Principle and Kuhn-Tucker conditions were applied. The model may be extended to a general model in which coefficients of state variables and control function in objective function and state equation are fuzzy random variables. Meanwhile, solving the obtained crisp model by the numerical methods such as wavelets may be considered for the future research problems.

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