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Motion Of Test Particle In The Space Time Of Black Hole In Conformal Weyl Gravity

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In this paper we consider the motion of test particles in the black hole space-time given by P. D. Mennheim and D. Kazanas. We derive the analytical solutions for the equation of motion of neutral test particles. the geodesic equations can be solved in terms of Weierstrass elliptic functions and derivatives of Kleinian sigma functions. the different types of the resulting orbits are characterized in terms of the conserved energy and angular momentum.

I. INTRODUCTION

One of such alternative theories of gravity is Conformal Gravity(CG)(Maldacena, 1997), a gravitational theory which is based on a large symmetry principle known as conformal symmetry. Intuitively, beside of local Lorentz symmetry, it also has an scaling symmetry in which the physics is invariant under the rescaling the metric as $g_{\mu\nu} = e^{\Omega(x)} g_{\mu\nu}$. The motion of test particles (both massive and massless) provides the only experimentally feasible way to study the gravitational fields of objects such as black holes.For this purpose the Weierstrassian elliptic functions are most useful because they lead to simple expressions. The resulting structure of the equations of motion is essentially the same as in Schwarzschild spacetime, where they can be solved analytically in terms of elliptic functions as first demonstrated by Hagihara in 1931 [8]. This method was also applied to the analytical solution of the equations of motion in 4-dimensional Schwarzschild de Sitter [7] and Kerr de Sitter spacetime [9], as well as in higher dimensional Schwarzschild, Schwarzschild-(anti)de Sitter, Reissner-Nordstrom 2 and Reissner-Nordstrom -(anti) de Sitter spactime [10]and in higher dimensional Myers-Perry spacetime [11].

II. METRIC AND FEILD EQUATION

Let us consider a conformal Weyl gravity. An exact static, spherically symmetric black hole solution is given by [1]

$$ds^2 = -B(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

where the coordinates are defined in the range $-\infty < t < \infty$, $r \geq 0, 0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$, and the lapse function, $B(r)$, is given by

$$B(r) = 1 - \frac{\beta(2-3\beta\gamma)}{r} - 3\beta\gamma + \gamma r - kr^2 \tag{2}$$

Here , k and are positive constants associated to the central mass, cosmological constant and the measurements of the departure of the Weyl theory from the Einstein - de Sitter, respectively. [2]

III. ANALYTICAL SOLUTION OF GEODESIC EQUATIONS

The geodesic motion in such a space-time is described by the geodesic equation

$$\frac{d^{x\mu}}{ds^2} + \Gamma^{\mu}_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} \tag{3}$$

where $\Gamma^{\mu}_{\rho\sigma}$ is the Christoffel symbol. The first constant of motion is given by the normalization condition $ds^2 = \frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{1}{2}\epsilon$ where for massive particles $\epsilon = 1$ and for light $\epsilon = 0$. conserved energy and angular momentum

$$E = g_{tt}\frac{dt}{ds} = \frac{dt}{ds}(1 - \beta\frac{(2-3\gamma\beta)}{r} - 3\beta\gamma + \gamma r - kr^2) \tag{4}$$

$$L = g_{\varphi\varphi}\frac{d\varphi}{ds} = r^2\frac{d\varphi}{ds} \tag{5}$$

which reduce the geodesic equation to one ordinary differential equation

$$\frac{dr}{d\tau} = E^2 - B(r)(\epsilon + \frac{L^2}{r^2}) \tag{6}$$

Together with energy and angular momentum conservation we obtain the corresponding equations for r as functions of φ

$$\frac{dr}{d\varphi} = \frac{r^4}{L^2}(E^2 - B(r)(\epsilon + \frac{L^2}{r^2})) \tag{7}$$

Eq.(6) gives a complete description of the dynamics of the geodesic motion. Eq.(6) suggests the introduction of an effective potential

$$V_{eff} = (1 - \frac{\beta(2-3\gamma\beta)}{r} - 3\beta\gamma + \gamma r - kr^2)(\epsilon + \frac{L^2}{r^2}) \tag{8}$$

For the analysis of the dependence of the possible types of orbits on the parameters of the space-time and the test

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particle or light ray it is convenient to use dimensionless quantities. Thus, we introduce

$$\tilde{\beta} = \frac{\beta}{m}, \tilde{\gamma} = m\gamma, \tilde{r} = \frac{r}{m}, \tilde{k} = m^2 k \quad (9)$$

and rewrite Eq.(7) as

$$(d\tilde{r}/d\varphi)^2 = k\epsilon\mathcal{L}\tilde{r}^6 - \gamma\epsilon\mathcal{L}\tilde{r}^5 + (E^2\mathcal{L} + 3\beta\gamma\epsilon\mathcal{L} - \epsilon\mathcal{L} + k)\tilde{r}^4 + (\epsilon\beta(2-3\beta\gamma)\mathcal{L} - \gamma)r^3 + (3\beta\gamma - 1)\tilde{r}^2 + \beta(2-3\gamma\beta)\tilde{r} = R_s(\tilde{r}) \quad (10)$$

The major point in this analysis is that Eq.(10) implies $R_s \geq 0$ as a necessary condition for the existence of a geodesic. Thus, the zeros of R_s are extremal values of $\tilde{r}(\varphi)$ and determine (together with the sign of R_s between two zeros) the type of geodesic. The polynomial R_s is in general of degree 6 and, therefore, has 6 (complex) zeros. but the positive real zeros are of interest for the type of orbit. As $\tilde{r} = 0$ is a zero of R_s for all values of the parameters, this zero is neglected in the following and

$$(d\tilde{r}/d\varphi)^2 = k\epsilon\mathcal{L}\tilde{r}^5 - \gamma\epsilon\mathcal{L}\tilde{r}^4 + (E^2\mathcal{L} + 3\beta\gamma\epsilon\mathcal{L} - \epsilon\mathcal{L} + k)\tilde{r}^3 + (\beta(2-\beta\gamma)\mathcal{L} - \gamma)r^2 + (3\beta\gamma - 1)\tilde{r} + \beta(2-\gamma\beta) = R_s(\tilde{r}) \quad (11)$$

is considered instead of R_s. By a comparison of coefficients we can solve the equations for E^2 and L dependent on ϵ.

$$L = \frac{(r-3\beta)(\gamma r - 3\gamma\beta + 2)}{r^2(-2kr^3 + \gamma r^2 - 3\gamma\beta^2 + 2\beta)} \quad (12)$$

$$E^2 = \frac{2(-kr^3 + \gamma r^2 - 3ar\beta + 3a\beta^2 + r - 2\beta)^2}{(r-3\beta)(\gamma r - 3\gamma\beta + 2)r} \quad (13)$$

Fig.(1) the results of this analysis are shown for test particles. we introduce a new variable $u = M/r$, which yields

$$(\frac{du}{d\varphi})^2 = (\beta(2-\gamma\beta))u^3 + (3\beta\gamma - 1)u^2 + (\beta(2-\beta\gamma)\mathcal{L} - \gamma)u + (E^2\mathcal{L} + 3\beta\gamma\epsilon\mathcal{L} - \epsilon\mathcal{L} + k) - \frac{\gamma\epsilon\mathcal{L}}{u} + \frac{k\epsilon\mathcal{L}}{u^2} \quad (14)$$

Null geodesics

For $\epsilon = 0$ Eq.(14) is of elliptic type $P_3(u) = \sum_{i=0}^{3} a_i u^i$. With the standard substitution $u = \frac{1}{a_3}(4y - \frac{a_2}{3})$ Eq.(14) can be transformed to the Weierstrass form and so that this equation turns into:

$$(\frac{dy}{d\varphi})^2 = 4y^3 - g_2 y - g_3 = P_3(y), \quad (15)$$

where $g_2 = \frac{a_2^2}{12} - \frac{a_1 a_3}{4}$ and $g_3 = \frac{a_1 a_2 a_3}{48} - \frac{a_0 a_3^2}{16} - \frac{a_2^3}{216}$ The analytical solution of Eq.(15) for $\epsilon = 0$ is then given by

$$y(\varphi) = \wp(\varphi - \varphi_{in}) \quad (16)$$

Then the solution of Eq.(10) acquires the form

$$\tilde{r}(\varphi) = \frac{a_3}{4\wp(\varphi - \varphi_{in}; g_2, g_3) - \frac{a_2}{3}}. \quad (17)$$

where

$$\varphi_{in} = \varphi_0 + \int_{y_0}^{\infty} \frac{dz}{\sqrt{4y^3 - g_2 - g_3}}, y_0 = \frac{a_3}{4\tilde{r_0}} + \frac{a_2}{12} \quad (18)$$

depends only on the initial values φ_0 and r_0.

Timelike geodesics

For $\epsilon = 1$ Eq. (14) should be rewritten as

$$(u\frac{du}{d\varphi})^2 = (\beta(2-\gamma\beta))u^5 + (3\beta\gamma - 1)u^4 + (\beta(2-\beta\gamma)\mathcal{L} - \gamma)u^3 + (E^2\mathcal{L} + 3\beta\gamma\epsilon\mathcal{L} - \epsilon\mathcal{L} + k)u^2 - \gamma\epsilon\mathcal{L}u + k\epsilon\mathcal{L} = p_5(u) \quad (19)$$

the solution of this equation is

$$u(\varphi) = -\frac{\sigma_1}{\sigma_2}\varphi(\sigma) \quad (20)$$

where σ_i is the i place derivative of Kleinian σ function and σ_z is

$$\sigma(z) = Ce^{zt}kz\theta[g,h](2\omega^{-1}z;\tau), \quad (21)$$

which is given by the Riemann θ-function with characteristic $[g,h]$. A number of parameters enters here: the symmetric Riemann matrix τ , the period-matrix$(2\omega, 2\acute{\omega})$, the periodmatrix of the second kind $(2\eta, 2\acute{\eta})$, the matrix$\kappa = \eta(2)^{-1}$ and the vector of Riemann constants with base point at innity $2[g,h] = (0,1)^t + (1,1)^t\tau$. The constant Ccan be given explicitly, see e.g. [6], but does not matter here.then the analytical solution of Eq.(10) is

$$\tilde{r}(\varphi) = -\frac{\sigma_2}{\sigma_1}\varphi_\sigma \quad (22)$$

Orbits

According to the Fig(1) and the Eqs.((12) , (13)) there are three regions. the physically acceptable regions are given by those values of r, for which $E^2 \geq V_{eff}$, The following different types of orbits can be identified

1. Flyby orbits: r starts from ∞, then approaches a periapsis $r = r_p$ and goes back to ∞.

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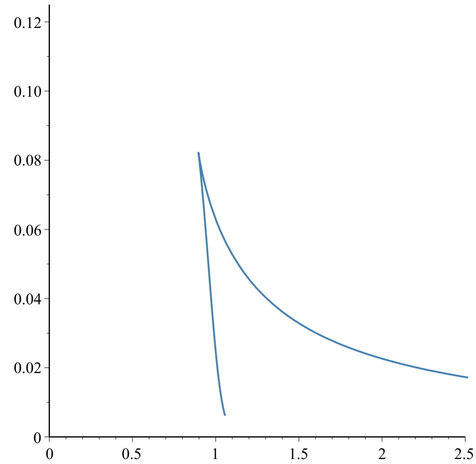

FIG. 1. different regions of geodesics movement of particles for values:$\epsilon = 1, k = \frac{1}{3.10^5}, \beta = 1, \alpha = 10^{-3}$.

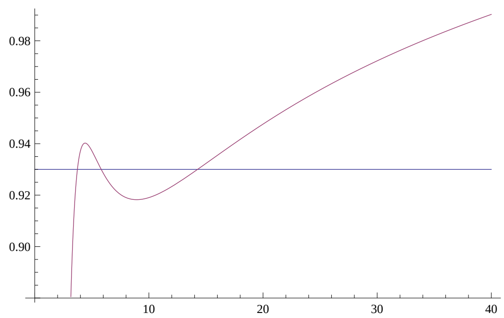

FIG. 2. effective potential of geodesics movement of particles. lateral line is second power of energy.

2. Bound orbits: r oscillates between two boundary values $r_p \leq r \leq r_a$ with $0 < r_p < r_a < \infty$.

3. Terminating bound orbits: r starts in $(0, r_a]$ for $0 < r_a < \infty$ and falls into the singularity at $r = 0$.

4. Terminating escape orbits: r comes from ∞ and falls into the singularity at $r = 0$.

The four regular types of geodesic motion correspond to different arrangements of the real and positive zeros of $R(r)$ defining the borders of $R(r) \geq 0$ or, equivalently, $E^2 \geq V_{eff}$. number of real and positive zeros of $R_s(\tilde{r})$ charaterize the possible orbits in every region. fore xample for the $E = \sqrt{0.94}, L = 0.07$ there is four real zeros, see (potential of Fig (2)) and that there are two orbits (Figs. (3 , 4):

bound orbit and terminating bound orbit which particle move from r_a and falling to singularity of black hole.

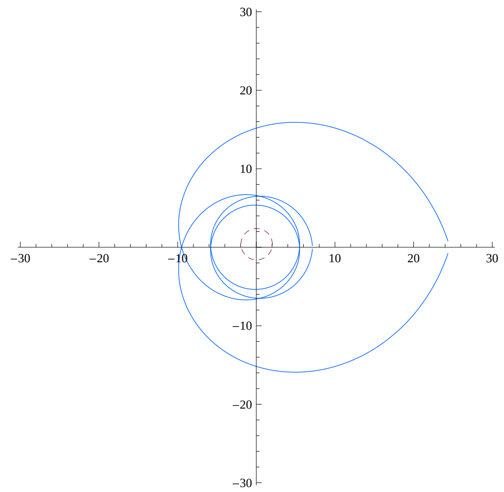

FIG. 3. $E^2 = \sqrt{0.94}, L = 0.07$.terminating bound orbit.

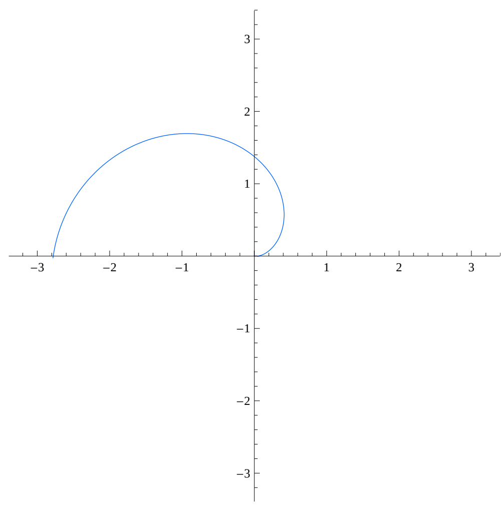

FIG. 4. $E^2 = \sqrt{0.94}, L = 0.07$.bound orbit.

IV. CONCLUSION

In this work we considered the motion of massive and masless test particles in the metric presented in [1].the geodesic equations can be solved in terms of Weierstrass elliptic functions and derivatives of Kleinian sigma functions. The results obtained in this paper can also present a usefull tool to calculate the exact orbits and their properties, including observables like the periastron shift of bound orbits, the light de ection of flyby orbits, the de ecton angle and the Lense-Thirring efect. It would be interesting to extend this work to a charged and rotating version.

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