Generalized second law of thermodynamics on the apparent horizon in $f(G)$-gravity

A. Abdolmaleki[1], T. Najafi[2],
[1]Center for Excellence in Astronomy & Astrophysics of Iran (CEAAI-RIAAM), Maragha, Iran
[2]Department of Physics, University of Kurdistan, Pasdaran St., Sanandaj, Iran

Here, we investigate the GSL of gravitational thermodynamics in the framework of modified Gauss-Bonnet gravity or $f(G)$-gravity. We consider a spatially FRW universe filled with the matter and radiation enclosed by the dynamical apparent horizon with the Hawking temperature. For two viable $f(G)$ models, we first numerically solve the set of differential equations governing the dynamics of $f(G)$-gravity. Then, we obtain the evolutions of the Hubble parameter, the Gauss-Bonnet curvature invariant term, the density and equation of state parameters as well as the deceleration parameter. Finally, we examine the validity of the GSL. For the selected $f(G)$ models, we conclude that both models have a stable de Sitter attractor. The equation of state parameters behave quite similar to those of the ΛCDM model in the radiation/matter dominated epochs, then they enter the phantom region before reaching the de Sitter attractor with $\omega = -1$. The deceleration parameter starts from the radiation/matter dominated eras, then transits from a cosmic deceleration to acceleration and finally approaches a de Sitter regime at late times, as expected. Furthermore, the GSL is respected for both models during the standard radiation/matter dominated epochs. Thereafter when the universe becomes accelerating, the GSL is violated in some ranges of scale factor. At late times, the evolution of the GSL predicts an adiabatic behavior for the accelerated expansion of the universe.

I. THE $F(G)$ THEORY OF GRAVITY

One of interesting alternative theories of gravity is modified Gauss-Bonnet gravity, so-called $f(G)$-gravity, where $f(G)$ is a general function of the Gauss-Bonnet curvature invariant term $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ [1–5]. The $f(G)$-gravity has recently been obtained a lot of tendency as a possible description of DE [6]. This kind of modified gravity (MG) theory has different particulars, among stability, the ability of description the present accelerated expansion of the universe, phantom divide line crossing and transition from deceleration to acceleration phases [7]. The action of modified Gauss-Bonnet gravity is given by [3]:

$$I = \int d^4x\sqrt{-g}\left(\frac{1}{2k^2}R + f(G) + L_r + L_m\right), \qquad (1)$$

where $k^2 = 8\pi G_N = 1$ and G_N is the Newtonian gravitational constant. Also g, R, L_r, L_m and $f(G)$ are the determinant of metric $g_{\mu\nu}$, Ricci scalar, the matter Lagrangian, the radiation Lagrangian and a general function of the Gauss-Bonnet term, respectively. For a spatially flat FRW universe, we have

$$R = 6\left(\dot{H} + 2H^2\right), \qquad G = 24H^2\left(\dot{H} + H^2\right), \qquad (2)$$

where $H = \dot{a}/a$ is the Hubble parameter and an overdot stands for a derivative with respect to the cosmic time t. Also, the Friedmann equations in $f(G)$-gravity take the form [8]

$$H^2 = \frac{1}{3}\rho_t, \qquad \dot{H} = -\frac{1}{2}(\rho_t + p_t). \qquad (3)$$

Here ρ_t and p_t are the total energy density and pressure defined as

$$\rho_t = \rho_m + \rho_r + \rho_G, \qquad p_t = p_m + p_r + p_G, \qquad (4)$$

where ρ_m and ρ_r are the energy density of matter and radiation, respectively. Also ρ_G and p_G are the energy density and pressure due to the $f(G)$ contribution defined as

$$\rho_G = Gf_G - f - 24H^3\dot{f}_G, \qquad (5)$$

$$p_G = 16H^3\dot{f}_G + 16H\dot{H}\dot{f}_G + 8H^2\ddot{f}_G - Gf_G + f. \qquad (6)$$

Moreover, the continuity equations governing the pressureless matter ($p_m = 0$), the radiation ($p_r = \rho_r/3$) and the $f(G)$ contribution are satisfied. By using of Eqs. (5) and (6), one can obtain the equation of state (EoS) parameter due to the $f(G)$ contribution as $\omega_G = p_G/\rho_G$. Also from Eq. (3), the effective EoS parameter can be obtained as $\omega_{eff} = p_t/\rho_t = -1 - \frac{2\dot{H}}{3H^2}$.

II. GSL IN $F(G)$-GRAVITY

As one of the most important theoretical touchstones to examine whether $f(G)$-gravity can be an alternative gravitational theory to general relativity, we explore the GSL of thermodynamics on the apparent horizon in $f(G)$-gravity, and obtain the condition for the GSL to be satisfied. The GSL says that the sum of horizon entropy and matter entropy inside the horizon must not decrease with time [9]. Now, we consider a spatially flat FRW universe filled with the matter and radiation. We further assume that the boundary of the universe to be enclosed by the dynamical apparent horizon with the Hawking temperature. The dynamical apparent horizon in a flat FRW universe is same as the Hubble horizon [10]. Following [9], the associated Hawking temperature on the apparent horizon $\tilde{r}_{\rm A}$ is given by

$$T_{\rm A}=\frac{1}{2\pi\tilde{r}_{\rm A}}\left(1-\frac{\dot{\tilde{r}}_{\rm A}}{2H\tilde{r}_{\rm A}}\right). \tag{7}$$

Now we are going to use the first law of thermodynamics to find the general condition needed to hold the GSL in $f(G)$-gravity. The entropy of matter and radiation inside the horizon are given by the Gibbs equation [11]

$$T_{\rm A}{\rm d}S_{\rm m}={\rm d}E_{\rm m}+p_{\rm m}{\rm d}V, \tag{8}$$

$$T_{\rm A}{\rm d}S_{\rm r}={\rm d}E_{\rm r}+p_{\rm r}{\rm d}V, \tag{9}$$

where $E_{\rm m}=\rho_{\rm m}V$ and $E_{\rm r}=\rho_{\rm r}V$. Also $V=4\pi\tilde{r}_{\rm A}^3/3$ is the volume of the dynamical apparent horizon $\tilde{r}_{\rm A}$ containing the pressureless matter ($p_{\rm m}=0$) and radiation ($p_{\rm r}=\rho_{\rm r}/3$).

Taking time derivative of Eqs. (8) and (9) and using the continuity equations one can get

$$T_{\rm A}\dot{S}=\frac{2\pi\left(\dot{H}+H^2\right)}{H^4}\Big[4\dot{H}-16H\left(H^2-2\dot{H}\right)\dot{f}_{\rm G}+16H^2\ddot{f}_{\rm G}\Big]. \tag{10}$$

where $S=S_{\rm r}+S_{\rm m}$. The horizon entropy in $f(G)$-gravity is given by [6]

$$S_{\rm A}=8\pi^2\left(H^{-2}+8f_{\rm G}\right). \tag{11}$$

Using Eqs. (7), (10) and (11), the GSL in $f(G)$-gravity yields

$$T_{\rm A}\dot{S}_{\rm tot}=\frac{2\pi}{H^4}\times\Big[2\dot{H}^2+8H\dot{H}\left(4\dot{H}+3H^2\right)\dot{f}_{\rm G}+16H^2\left(\dot{H}+H^2\right)\ddot{f}_{\rm G}\Big], \tag{12}$$

where $S_{\rm tot}=S_{\rm r}+S_{\rm m}+S_{\rm A}$. Equation (12) shows that in $f(G)$ gravity, the validity of the GSL, i.e. $T_{\rm A}\dot{S}_{\rm tot}\geq 0$, depends on the explicit form of the $f(G)$ model. For the Einstein gravity ($f(G)=0$), one can immediately find that the GSL (12) reduces to

$$T_{\rm A}\dot{S}_{\rm tot}=\frac{4\pi\dot{H}^2}{H^4}\geq 0, \tag{13}$$

which shows that the GSL is always fulfilled throughout history of the universe. In what follows, we examine the validity of the GSL (12) for two viable $f(G)$-gravity models.

III. TWO VIABLE $F(G)$ MODELS

Here, we are interested in examining the GSL for two viable $f(G)$ models which are introduced by [3,4] to explain the accelerated expansion of the universe at present. The first model has the form [3]

$$f(G)=\alpha\left(G^{\frac{3}{4}}-\beta\right)^{\frac{2}{3}},\qquad \text{Model I}, \tag{14}$$

where α and β are two constants of the model. The second $f(G)$ model is given by [4]

$$f(G)=\lambda\frac{G}{\sqrt{G_*}}\arctan\left(\frac{G}{G_*}\right)-\frac{\lambda}{2}\sqrt{G_*}\,\ln\left(1+\frac{G^2}{G_*^2}\right)-\alpha\lambda\sqrt{G_*},\qquad \text{Model II}, \tag{15}$$

where α is an arbitrary constant and λ is a positive constant. Also $G_*=H_0^4$ and H_0 is the Hubble parameter at present.

With choice of suitable initial conditions, we numerically solve the differential equations governing the dynamics of $f(G)$-gravity for both model I and model II [12]. The evolutions of the Hubble parameter H, the Gauss-Bonnet curvature invariant term $|G|$ and the quantity $H^6f_{\rm GG}$ versus $N=\ln(a/a_{\rm i})$ for model I and model II are plotted in Figs. 1 and 2, respectively. Figures show that: (i) the Hubble parameter decreases during history of the universe. (ii) The GB term switches its sign during the transition from the standard radiation/matter dominated epochs to the accelerated era (which corresponds to passing through the minus infinity in logarithmic scale). (iii) The quantity $H^6f_{\rm GG}$ satisfies the condition $0<H_1^6f_{\rm GG}(G_1)<1/384$ which shows that both models have a stable de Sitter attractor. (iv) H, $|G|$ and $H^6f_{\rm GG}$ at late times go to a constant value when the universe enters a de Sitter regime. Notice that the result of Fig. 2 for model II is the same as that obtained in [4].

The evolutions of the density parameters $\Omega_{\rm r}$, $\Omega_{\rm m}$, $\Omega_{\rm G}$ and the effective EoS parameter $\omega_{\rm eff}$, versus N for model I and model II are plotted in Figs. 3 and 4, respectively. Figures illustrate that: (i) for both models, $\Omega_{\rm r}$, $\Omega_{\rm m}$, $\Omega_{\rm G}$ and $\omega_{\rm eff}$ behave quite similar to those of the ΛCDM model in the radiation/matter dominated epochs. (ii) For model I, $\omega_{\rm eff}$ oscillates rapidly during the accelerated epoch and goes deep into the phantom-like region as the universe enters the de Sitter period. (iii) For model II, $\omega_{\rm eff}$ oscillates slowly around -1 as the system enters the epoch of cosmic acceleration, which implies that the de Sitter solution is a stable spiral. Note that the results of Figs. 3 and 4 are the same as those obtained in [3] and [4], respectively.

The evolution of the deceleration parameter $q=-1-\dot{H}/H^2$, for model I and model II is plotted in Figs. 5 and 6, respectively. Figure 5 clears that for model I, the deceleration parameter starts from $q=1$ corresponding to

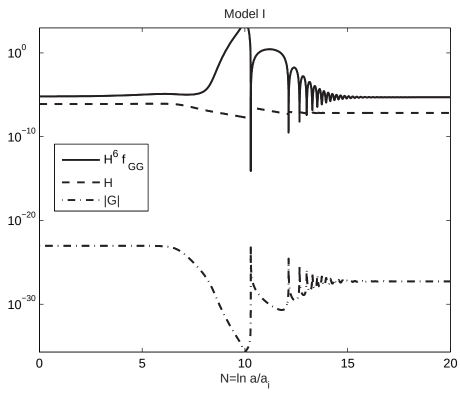

FIG. 1. The evolutions of the Hubble parameter H, the Gauss-Bonnet curvature invariant term $|G|$ and the quantity $H^6 f_{\rm GG}$, versus $N = \ln(a/a_{\rm i})$ where $a_{\rm i}$ is the initial value of the scale factor. Auxiliary parameters are: $\alpha = \frac{1}{40\sqrt{66}}$ and $\beta = -10^{-17}$.

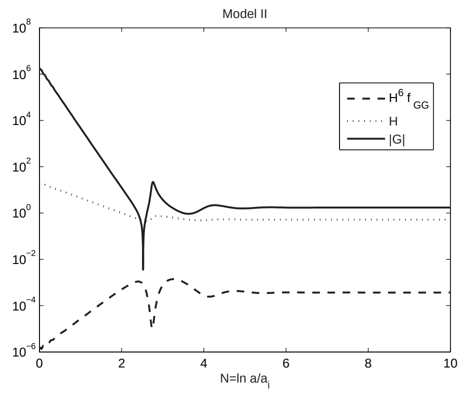

FIG. 2. Same as Fig. 1 but for model II. Auxiliary parameters are: $\alpha = 10$ and $\lambda = 0.075$.

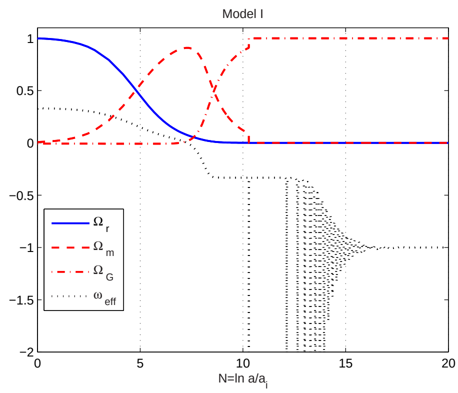

FIG. 3. The evolutions of $\Omega_{\rm m}$, $\Omega_{\rm G}$, $\Omega_{\rm r}$ and $\omega_{\rm eff}$, versus N. Auxiliary parameters as in Fig. 1.

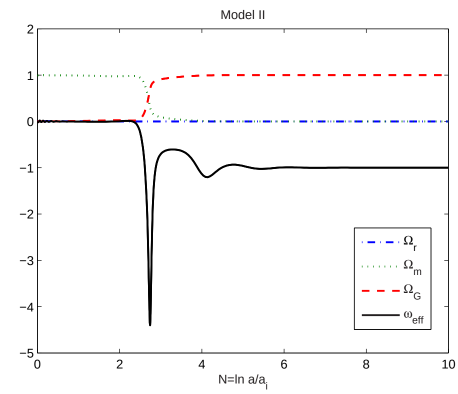

FIG. 4. Same as Fig. 3 but for model II. Auxiliary parameters as in Fig. 2.

the radiation dominated epoch, then shows a cosmic deceleration ($q > 0$) to acceleration ($q < 0$) transition [13] and finally oscillates rapidly into the de Sitter regime ($q = -1$). Figure 6 presents that for model II, q varies from the matter dominated epoch ($q = 0.5$), then transits from a cosmic deceleration to acceleration and approaches smoothly a de Sitter regime at late times, as expected.

Finally, we examine the validity of the GSL for both models. In Figs. 7 and 8, we plot the variation of the GSL, Eq. (12), versus N for model I and model II, respectively. Figures illustrate that for both models, the GSL during the radiation/matter dominated epochs is fulfilled. Thereafter when the universe enters the cosmic acceleration era, i.e. $q < 0$ see Figs. 5 and 6, the GSL does not hold (i.e. $T_{\rm A}\dot{S}_{\rm tot} < 0$) in some ranges of N. At late times, the GSL for model I oscillates rapidly and for model II approaches smoothly into the de Sitter universe, adiabatically (i.e. $T_A\dot{S}_{\rm tot} = 0$).

IV. CONCLUSIONS

One of gravitational alternative theories for DE is $f(G)$-gravity, in which DE emerges from the modification of the Gauss-Bonnet invariant term. Here, we investigated the GSL of gravitational thermodynamics in the framework of $f(G)$-gravity. To do so, we considered a spatially flat FRW universe filled with the pressureless matter and radiation. We supposed the boundary of the universe to be enclosed by the dynamical apparent horizon with the Hawking radiation. We derived a general relation for the GSL which its validity depends on $f(G)$-model. Hence, for two viable $f(G)$-models, we first solved numerically the set of differential equations governing the dynamics of $f(G)$-gravity. Then, we examined the validity of the GSL for the two selected $f(G)$-models. Our results show that the GSL is fulfilled for both models during the standard radiation/matter dominated epochs. But when the universe becomes accelerating, the GSL is violated (i.e. $T_{\rm A}\dot{S}_{\rm tot} < 0$) in some ranges of scale factor. At late times, the evolution of the GSL predicts an

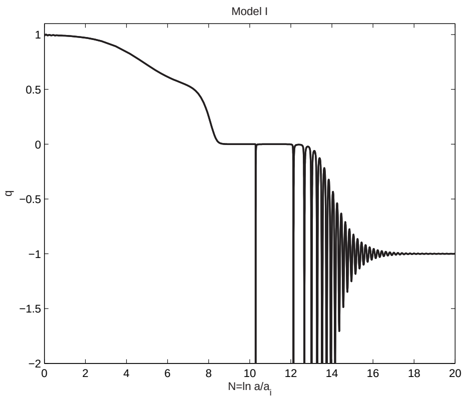

FIG. 5. The evolution of the deceleration parameter q versus N for model I. Auxiliary parameters as in Fig. 1.

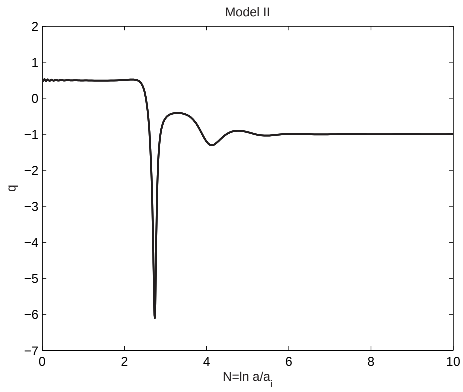

FIG. 6. Same as Fig. 5 but for model II. Auxiliary parameters as in Fig. 2.

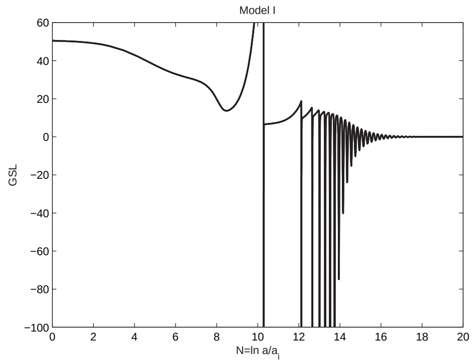

FIG. 7. The evolution of the GSL versus N for model I. Auxiliary parameters as in Fig. 1.

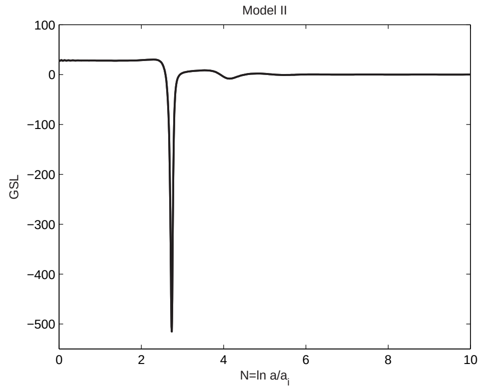

FIG. 8. Same as Fig. 7 but for model II. Auxiliary parameters as in Fig. 2.

adiabatic behavior (i.e. $T_A \dot{S}_{\rm tot} = 0$) for the accelerated expansion of the universe.

---

[1] S. Nojiri, S.D. Odintsov, Phys. Lett. B **631**, 1 (2005).
[2] B. Li, J.D. Barrow, D.F. Mota, Phys. Rev. D **76**, 044027 (2007);
S.C. Davis, Prog. Theor. Phys. Suppl. **172**, 81 (2008).
[3] S.Y. Zhou, E.J. Copeland, P.M. Saffin, JCAP **07**, 009 (2009).
[4] A. De Felice, S. Tsujikawa, Phys. Lett. B **675**, 1 (2009).
[5] A. De Felice, S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).
[6] R.M. Wald, Phys. Rev. D **48**, 3427 (1993);
S.M. Carroll, et al., Phys. Rev. D **71**, 063513 (2005);
G. Cognola, et al., JCAP **02**, 010 (2005).
[7] M. Alimohammadi, A. Ghalee, Phys. Rev. D **79**, 063006 (2009).
[8] S. Capozziello, et al., Int. J. Mod. Phys. D **12**, 1969 (2003);
S. Capozziello, V.F. Cardone, A. Troisi, Phys. Rev. D **71**, 043503 (2005).
[9] R.G. Cai, S.P. Kim, JHEP **02**, 050 (2005).
[10] E. Poisson, W. Israel, Phys. Rev. D **41**, 1796 (1990);
S.A. Hayward, Phys. Rev. D **53**, 1938 (1996);
Y.G. Gong, A. Wang, Phys. Rev. Lett. **99**, 211301 (2007).
[11] G. Izquierdo, D. Pavón, Phys. Lett. B **639**, 1 (2006).
[12] A. Abdolmaleki, T. Najafi, arXiv:1504.03544.
[13] E.E. Ishida, et al., Astropart. Phys. **28**, 547 (2008).