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A novel methodology for optimal control problems with application to coordinate supplier development in a two-echelon supply chain

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Abstract
This paper deals with coordinating supplier development programs in a two-echelon supply chain which is formulated as a continuous time optimal control model. Drawing upon advanced ingredients of differential and Poisson geometry, a novel methodology is presented for the optimal control problem by reformulating and converting the Hamilton-Jacobi-Bellman partial differential equation (PDE) to a reduced Hamiltonian system, so that the exact optimal solution of the control problem can be obtained, instead of numerical estimation. The proposed methodology is applied to the problem of coordinating supplier development in a supplier-manufacturer supply chain to find the exact optimal solution. The analytical solution to the problem is obtained based on the proposed method and a numerical example is presented to further validate its applicability and superiority. The proposed methodology can be also applied to control problems in other optimization fields.

Keywords: Optimal control problem, Poisson bracket, Hamiltonian system, First integral, Supplier development, Supply chain coordination

Introduction
It is known that strong supply chain relationship is crucial for improving operational efficiency and developing sustainable competitive advantage [1]. For a manufacturing firm, closer relationship with its suppliers enhances profitability of both the manufacturer and the suppliers [2].
In many industries, manufacturing firms develop strategic, long-term relationships with their suppliers by implementing and supporting supplier development programs. The goal is to improve the performance and capabilities of the suppliers which, in turn, results in improving operational performance in terms of cost, quality, delivery, etc. [2-4].
Despite the potential benefits of supplier development programs, they might be unattractive for the suppliers since the suppliers might be reluctant to modify their internal processes and instead pursue their own objectives [5]. Because the success of supplier development program depends on mutual recognition and aligned objectives, coordination between the supplier and manufacturer is required [2, 6].
In recent years, the subject of supply chain coordination has received much attention in the literature (e.g. [7, 8]). It deals with making globally optimal supply chain decisions that can benefit all the parties involved, instead of individual decisions, in order to improve the overall performance and efficiency of the supply chain. Various mechanisms are used for coordination purposes [9, 10].
In many situations, the problem of supply chain coordination is formulated as a continuous time optimal control model with an equation of incomplete Hamiltonian system in which the optimal solution should be estimated by numerical analysis. This paper presents a novel methodology based on differential and Poisson geometry by reformulating and converting the original problem to a reduced Hamiltonian system. Therefore, the exact solution to the problem can be obtained. In order to illustrate applicability and superiority of the proposed methodology, it is applied to obtain the optimal solution to the problem of coordinating supplier development in a two-echelon supply chain comprising of a single supplier and single manufacturer.
First, in preliminaries section, we study some geometric ingredients which enable us to extend the optimal control problem to include more variables and more equations in $\Re^m$ by considering the other derivations of the Hamiltonian function. In fact, by dynamical programming, we can write down an infinitesimal version of the optimal control problem as a partial differential equation (PDE), i.e. the Hamilton-Jacobi-Bellman PDE. Then we consider the Hamiltonian system corresponded to this PDE.
On the other hand, for a Hamiltonian system there is a connection between one–parameter variational symmetry groups of the system and first integrals. In Theorem 3 expressed in notions of first integrals and generalized symmetry groups, we prove that the problem for each

admissible control can be solved in a reduced Hamiltonian system corresponded to Hamilton–Jacobi–Bellman PDE on (nonlinear) control problem. Then we solve an example by reduction method. Specially, we apply this method to our models about coordination of a supplier-manufacturer.

Note that the use of symmetry groups to effect a reduction in order of a Hamiltonian system of ordinary differential equations parallels the methods for Euler–Lagrange equations, but with the added advantages of an immediate geometrical interpretation. In this way the optimal control problem with dynamical demand which applying Pontryaginn's Maximum principle to corresponding Lagrangian system (e.g., in [11]) can be extended to this framework.

Preliminaries

Optimal Control problem

We consider the simplest form of the optimal control problem as

$$\begin{aligned}&\max J=\int_0^T k(t,x,u)dt,\\ &\text{subject to } \dot{x}=f(t,x,u), \qquad (1)\\ &x(0)=A,\ x(T)\ \text{free},\ (A,\ T\ \text{given})\\ &\text{and } u(t)\in U,\ \forall t\in[0,T]\end{aligned}$$

where U denoted some bounded control set, t, time, x, state, and u as control. Also, another variable $\lambda=\lambda(t)$ is the costate variable (or auxiliary variable) which will emerge in the solution process, by Hamiltonian function, H, which is defined as

$$H(t,x,u,\lambda)=k(t,x,u)+\lambda(t)f(t,x,u)\cdot \qquad (2)$$

For the problem in (1), and with the Hamiltonian defined in (2), the maximum principle conditions are

$$\begin{aligned}&\max_u H(t,x,u,\lambda),\ \text{for all } t\in[0,T],\\ &\dot{x}=\frac{\partial H}{\partial\lambda},\ (\text{equation of motion for } x) \qquad (3)\\ &\lambda=-\frac{\partial H}{\partial x},\ (\text{transversality condition})\end{aligned}$$

where the symbol $\max_u H$ means that the Hamiltonian is to be maximized with respect to u alone as the choice variable and we obtain to a Hamiltonian system.

Note that we shall deal exclusively with the maximization problem in control theory. In this way, the necessary conditions for optimization can be stated with more specificity and less confusion. In situations where a minimization problem is encountered, such as in the sequel, it can be always reformulated in the maximization form by simply attaching a minus sign to the objective function.

Now, we study some geometric notions which are needed in the sequel. More details can be found in [12].

In two following sections, we give only a summary of some notions from differential and Poisson geometry to establish the main result and to make the paper essentially self-contained.

Geometric ingredients

Preliminaries of differential geometry

As the notions of [13] or [14], we consider a m-dimensional manifold which defined as a set M, together with a countable coordinate charts $U_\alpha\subset M$ and one–to–one local coordinate maps $\varphi_\alpha:U_\alpha\to V_\alpha$ onto connected open subsets $V_\alpha\subset\Re^m$, which satisfy the following properties:

The coordinated charts cover M. On the overlap of any pair of coordinate charts $U_\alpha\bigcap U_\beta$ the composite map $\varphi_\beta\circ\varphi_\alpha^{-1}:\varphi_\alpha(U_\alpha\bigcap U_\phi)\to\varphi_\beta(U_\alpha\bigcap U_\beta)$, is a smooth function. If $x\in U_\alpha$, $\tilde{x}\in U_\beta$ are distinct points of M, then, there exist open subsets $W\subset V_\alpha$, $\tilde{W}\subset V_\beta$, with $\chi_\alpha(x)\in W$, $\chi_\beta(\tilde{x})\in\tilde{W}$ and $\varphi_\alpha(W)\bigcap\varphi_\beta(W)$ is empty set.

Suppose γ is a smooth curve on a manifold M, parameterized by $\gamma:I\to M$, where I is a subinterval of $\Re$. At each point of γ the curve has a tangent vector $\dot{\gamma}=d\gamma/dt=(\dot{\gamma}_1,\ldots,\dot{\gamma}_m)$. The tangent space to M at x, denoted by $TM|_x$, is an m–dimensional vector space, with a basis $\{\partial/\partial x_1,\ldots,\partial/\partial x_m\}$ in the given local coordinates. The integral curve of a vector field $\vec{v}|_x\in TM|_x$ is a smooth parameterized curve $x=\gamma(t)$ whose tangent vector at any point coincides with the value of $\vec{v}$ at the same point $\gamma(t)=\vec{v}|_{\gamma(t)}$ for all t.

If $\vec{v}$ is a vector field, we denote the parameterized maximal integral curve passing through x in M by $\Phi(t,x)$ named as the flow generated by $\vec{v}$ or a one–parameter group of transformations and the vector field $\vec{v}$ is called the infinitesimal generator of the action. There is a one–to–one correspondence between local one–parameter groups of transformations and their infinitesimal generators.

For computation of the one–parameter group generated by a given vector field $\vec{v}$ we refer to as exponentiation of the vector field $\exp(t\vec{v})x\equiv\Phi(t,x)$ which for all $x\in M$

$$\frac{d}{dt}[\exp(t\vec{v})x]=v|_{\exp(t\vec{v})x}, \qquad (4)$$

If $\vec{v}=\sum_i\xi_i(x)\partial/\partial\xi_i$ and $f:M\to\Re$ a smooth function, then using the chain rule and (4) we find

$$\begin{aligned}&\frac{d}{dt}f(\exp(t\vec{v})x)=\sum_{i=1}^m\xi(\exp(t\vec{v})x)\frac{\partial f}{\partial x_i}(\exp(t\vec{v})x) \qquad (5)\\ &=\vec{v}(f)[\exp(t\vec{v})(x)].\end{aligned}$$

For a smooth real–valued function $f(x)=f(x_1,\ldots,x_r)$ of r independent variables, there are $r_k\equiv\binom{r+k-1}{k}$, different k–th order partial derivatives of f. We employ the multi–

index notation $\partial f(x)=\partial^k f(x)/\partial x_{j_1}\partial x_{j_2}...\partial x_{j_k}$, for these derivatives. More generally, if $f: X \to U$ is a smooth function from $X \approx \Re^r$ to $U \approx \Re^s$, so $u=f(x)=(f_1(x),..., f_s(x))$, there are $s.r_k$ numbers $u_J^\alpha=\partial_J f_\alpha(x)$ needed to represent all the different k–th order derivatives of the components of f at a point x. We let $U_k \equiv \Re^{s.r_k}$ be the Euclidean space of this dimension, endowed with coordinates u_J^α corresponding to $\alpha=1,...,s$, and all multi–indices $J=(j_1,..., j_k)$ of order k, designed so as to represent the above derivatives. Furthermore, $U^{(n)}=U\times U_1\times...\times U_n$ is the Cartesian product space, whose coordinates represent all the derivatives of functions $u=f(x)$ of all orders from 0 to n. A typical point in $U^{(n)}$ will be denoted by $u^{(n)}$.

We consider the n–th prolongation of a smooth function $u=f(x)$, $f: X\to U$, i.e., $u^{(n)}=pr^{(n)}f(x)$ which is defined by the equations $u_J^\alpha=\partial_J f_\alpha(x)$. Furthermore, we consider a system of n–th order differential equations in r independent and s dependent variables which given as a system of equations $\Delta_\rho(x,u^n)=0$, for $\rho=1,...,l$, involving $x=(x_1,...,x_r)$, $u=(u_1,...,u_s)$ and the derivatives of u with respect to x up to order n. For a vector field $\vec{v}$ on $M\subset X\times U$ the n–th prolongation of $\vec{v}$, denoted $pr^{(n)}\vec{v}$, will be a vector field on the n–jet space $M^{(n)}$, and is defined to be the infinitesimal generator of the corresponding prolonged one–parameter group $pr^{(n)}[\exp(t\vec{v})]= d/dt\,|_{t=0}\; pr^{(n)}[\exp(t\vec{v})(x,u^{(n)})]$, for any $(x,u^{(n)})\in M^{(n)}$.

Let $P(x,u^{(n)})$ be a smooth function of x, u and derivatives of u up to order n, defined on an open subset $M^{(n)}\subset X\times U^{(n)}$. The total derivative of P with respect to x_i is the unique smooth function $D_i P(x,pr^{(n+1)}f(x))=\partial/\partial x_i\left\{P(x,pr^{(n)}f(x))\right\}$ defined on $M^{(n+1)}$. We can show that for $P(x,u^{(n)})$, the i–th total derivative of P has the general form

$$D_i P=\frac{\partial P}{\partial x_i}+\sum_{\alpha=1}^{s}\sum_J u_{J,i}^\alpha\frac{\partial P}{\partial u_J^\alpha}, \tag{6}$$

where, for $J=(j_1,..., j_k)$,

$$u_{J,i}^\alpha=\frac{\partial u_J^\alpha}{\partial x_i}=\frac{\partial^{k+1}u^\alpha}{\partial x_i\partial x_{j_1}...\partial x_{j_k}}. \tag{7}$$

In (6), the sum is over all J's of order $0\le \# J\le n$, where n is the highest order derivative appearing in P. For a vector field $\vec{v}=\sum_{i=1}^{r}\xi_i(x,u)\partial/\partial x_i+\sum_{\alpha=1}^{s}\phi_\alpha(x,u)\partial/\partial u_\alpha$ on an open subset $M\subset X\times U$, the n–th prolongation of $\vec{v}$ is the vector field $pr^{(n)}\vec{v}=\vec{v}+\sum_{\alpha=1}^{s}\sum_J\phi_\alpha^J(x,u^{(n)})\partial/\partial u_J^\alpha$, defined on the corresponding jet space $M^{(n)}\subset X\times U^{(n)}$. The coefficient functions ϕ_α^J of $pr^{(n)}\vec{v}$ are given by the formula $\phi_\alpha^J(x,u^{(n)})=D_J(\phi_\alpha-\sum_{i=1}^{r}\xi_i u_i^\alpha)+\sum_{i=1}^{r}\xi_i u_{J,i}^\alpha$, where of (9) we have $u_i^\alpha=\partial u_\alpha/\partial x_i$, and $u_{J,i}^\alpha=\partial u_J^\alpha/\partial x_i$.

A generalized vector field will be a (formal) expression of the form $v=\sum_{i=1}^{r}\xi_i[u]\partial/\partial x_i+\sum_{\alpha=1}^{s}\phi_\alpha[u]\partial/\partial u_\alpha$ in which ξ_i and ϕ_α are smooth differential functions, $\xi_i[u]=\xi_i(x,u^{(n)})$ and $\phi_\alpha[u]=\phi_\alpha(x,u^{(\alpha)})$, too.

Note. We can show that a generalized vector field $\vec{v}$ is a generalized infinitesimal symmetry of a system of differential equations

$$\Delta_\rho[u]=\Delta_\rho(x,u^{(n)})=0,\quad \rho=1,...,l, \tag{8}$$

if and only if

$$pr\vec{v}[\Delta_\rho]=0,\quad \rho=1,...,l, \tag{9}$$

for every smooth solution $u=f(x)$.

Preliminaries of Poisson geometry

As notions of [15] or [16], we consider a manifold M with a Poisson structure $\{.,.\}$ on M which is an operation that assigns a smooth real–valued function $\{F,G\}$ on M to each pair F,G of smooth, real–valued functions, with the basic properties of bilinearity, skew–symmetry, Jacobi Identity and Leibniz' rule.

Let M be a Poisson manifold and $H: M\to\Re$ a smooth function. The Hamiltonian vector field associated with H is the unique smooth vector field $\hat{v}_H$ on M satisfying $\hat{v}_H(F)=\{F,H\}=-\{H,F\}$. In local coordinates $x=(x_1,...,x_m)$ on M, the associated Hamiltonian vector field will be of the general form $v_H=\sum_{i=1}^{m}\xi_i(x)\partial/\partial x_i$, where the coefficient functions $\xi_i(x)$, which depend on H, are to be determined. Then we have

$$\{F,H\}=\sum_{i=1}^{m}\{x_i,H\}\frac{\partial F}{\partial x_i}. \tag{10}$$

Using the slew–symmetry of the Poisson bracket, and (10) we obtain the basic formula

$$\{F,H\}=\sum_{i=1}^{m}\sum_{j=1}^{m}\{x_i,x_j\}\frac{\partial F}{\partial x_i}\frac{\partial H}{\partial x_j}, \tag{11}$$

for the Poisson bracket. We assemble the structure functions of the Poisson manifold M relative to the given local coordinates, $J^{ij}(x)=\{x_i,x_j\}$ for $i,j=1,...,m$ into a skew–

symmetric $m \times m$ structure matrix $J(x)$ of M. Using (11) the Hamiltonian vector field associated with $H(x)$ has the form

$$\hat{v}_H = \sum_{i=1}^{m}\left(\sum_{j=1}^{m} J^{ij}(x)\frac{\partial H}{\partial x_j}\frac{\partial}{\partial x_i}\right) \quad (12)$$

Therefore, in the given coordinate chart, Hamilton's equations take the form

$$\frac{dx}{dt} = J(x)\nabla H(x)\cdot \quad (13)$$

For example, in the manifold $M = \Re^{2n}$ with coordinates $(p,q) = (p_1,...,p_n,q_1,...,q_n)$, if $F(p,q)$ and $G(p,q)$ are smooth functions, we define their Poisson bracket to be the function

$$\{F,G\} = \sum_{i=1}^{n}\left\{\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}\right\}. \quad (14)$$

In the case of standard bracket (14), as in equation (11), the Hamiltonian vector field corresponding to $H(p,q)$ is $\hat{v}_H = \sum_{i=1}^{n}\left(\partial H/\partial p_i\, \partial/\partial q_i - \partial H/\partial q_i\, \partial/\partial p_i\right)$. The corresponding flow is obtained by integrating the system of ordinary differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad i = 1,...,n, \quad (15)$$

which are Hamilton's equations in this case. More concepts can be found in [15].

Main acquirements of Poisson and differential ingredients from before two last elementary section we conclude that:

Attainment 1. If $\hat{v}_\psi$ be the Hamiltonian vector field determining (12), by (5), for any solution to Hamiltonian equations $d\psi(x(t),t)/dt = \partial\psi/\partial t(x(t),t) + \hat{v}_H(\psi)(x(t),t)$, we have $d\psi/dt = 0$ along solutions if and only if

$$\frac{\partial\psi}{\partial t} + \{\psi,H\} = 0, \quad (16)$$

holds everywhere. Thus, a function $\psi(x,t)$ is a first integral for the Hamiltonian system (13) if and only if (16) holds for all x,t.

Attainment 2. If $\psi(x,t)$ be a first integral of a Hamiltonian system, then, the Hamiltonian vector field $\hat{v}_\psi$ determined by ψ, as in equation (12), generates a one–parameter symmetry group of the system. This means that the Hamiltonian vector field is an infinitesimal generator (in evolutionary form) of a one–parameter group of transformations acting on an open set of the space of independent and dependent variables for the system which are invariant under the element of the group. Then, by (8) and (9), the vector field $\vec{v} = v_H = \psi(t,x,\partial u/\partial x)\partial_u$ is a generalized symmetry of the Hamilton-Jacobi equation if and only if $\psi(t,p,q)$ is a first integral of Hamilton's equations.

Attainment 3. For Hamiltonian systems, one–parameter Hamiltonian symmetry groups whose infinitesimal generators are Hamiltonian vector fields arise from variational symmetry groups. The use of symmetry groups to effect a reduction in order of a Hamiltonian system of ordinary differential equations is the main topic of our methodology. By notions of before sections, we have

Reduction Theorem 1. Suppose $\hat{v}_\psi \neq 0$ generates a Hamiltonian symmetry group of the Hamiltonian system $\dot{x} = J\nabla H$ corresponding to the time–independent first integral $\psi(x)$. Then, there is a reduced Hamiltonian system involving two fewer variables with the property that every solution of the original system can be determined using a single quadrature from those of the reduced system [13].

Reduction Theorem 2. Let $\dot{x} = J\nabla H$ be a Hamiltonian system in which $h(x)$ does not depend on t. Then, there is a reduced, time–dependent Hamiltonian system in two fewer variables, from whose solutions those of the original system can be found by quadrature [13].

Proposed methodology

We study the possibility of optimally controlling the solution $\vec{x}(.)$ of the ordinary differential equation

$$\begin{aligned} &\dot{x}(t) = f(\vec{x}(s),\vec{\alpha}(s)), \quad (t < s < T) \\ &\vec{x}(t) = x\cdot \end{aligned} \quad (17)$$

Here $\cdot = d/dt$, $T > 0$ is a fixed terminal time, and $x \in \Re^n$ is a given initial point, taken on by our solution $\vec{x}(.)$ at the starting time $t \geq 0$. At later times $t < s < T$, $\vec{x}(.)$ evolves according to the ODE, where $f: \Re^n \times A \to \Re^n$ is a given bounded, Lipchitz continuous function, and A is some given compact subset of, say, $\Re^m$. The function $\vec{\alpha}(.)$ appearing (17) is a control, that is, some appropriate scheme for adjusting parameters from the set A as time evolves, thereby affecting the dynamics of the system modeled by (17). For more details refer to [17].

Let $A := \{\vec{\alpha}:[0,T] \to A \mid \vec{\alpha}(.) \text{ is measurable}\}$, the set of admissible controls. Our goal is to find a control $\vec{\alpha}^*(.)$ which optimally steers the system. For this purpose, given $x \in \Re^n$ and $0 \leq t \leq T$, let us define for each admissible control $\vec{\alpha}(.) \in \mathrm{A}$ the corresponding cost

$$c_{x,t}[\vec{\alpha}(.)] := \int_t^T k(\vec{x}(s),\vec{\alpha}(s))ds + g(\vec{x}(T)), \quad (18)$$

where $\vec{x}(.) = \vec{x}^{\vec{\alpha}(.)}(.)$ solves the ODE (20) and $k: \Re^n \times A \to \Re$, $g: \Re^n \to \Re$ are given functions, k is the running cost per unit time and g is the terminal cost. Now, we can express our main results:

Theorem 3. (Reduced control problem) We can solve the optimal problem (17) of the cost function (18) for each admissible control $\vec{\alpha}(.) \in \mathrm{A}$, as the above notions, in a reduced Hamiltonian system involving fewer variables.

Proof. The method of dynamic programming investigates the above problem by turning attention to the value function $u(x,t):=\inf_{\vec{\alpha}(.)\in A} c_{x,t}[\vec{\alpha}(.)], x\in\Re^n \quad ,0\le t\le T$. For each $\tau>0$ so small that $t+\tau\le T$, we have

$$u(x,t)=\inf_{\vec{\alpha}(.)\in A}\left\{\int_t^{t+\tau} k(\vec{x}(s),\vec{\alpha}(s))ds+u(\vec{x}(t+\tau),t+\tau)\right\}, \qquad (19)$$

where $\vec{x}(.)=\vec{x}^{\vec{\alpha}(.)}(.)$ solves the ODE (17) for the control $\vec{\alpha}(.)$. We can write down as a PDE an infinitesimal version of the optimality conditions (19). The value function u is the unique viscosity solution of this terminal-value problem for the Hamilton-Jacobi-Bellman equation:

$$u_t+\min_{a\in A}\{f(x,a).Du+k(x,a)\}=0, \ in \ \Re^n\times(0,T)$$
$$u=g \ on \ \Re^n\times\{t=T\}, \qquad (20)$$

where $Du=D_x u=(u_{x_1},...,u_{x_n})$. In similar way to problem (1) with Hamiltonian (2), the Hamilton-Jacobi-Bellman PDE (20) has the form $u_t+H(Du,x)=0 \ in \ \Re^n\times(0,T)$, for the Hamiltonian

$$H(p,x):=\min_{a\in A}\{f(x,a).p+k(x,a)\}, \qquad (21)$$

for $p,x\in\Re^n$, where p is the name of the variable for which we substitute the gradient Du in the PDE. The corresponding to a Hamiltonian system, as in equation (15), is the Hamilton-Jacobi partial differential equation $\partial u/\partial t+H(\partial u/\partial x,x,t)=0$. Note that the maximum principle condition mentioned in (3) is a simplified version of this equation. Now, we can apply Attainments 1-3, step by step, to this Hamiltonian system. Then of Reduction Theorems 1 and 2, the knowledge of first integrals allows us to reduce the order of the system. The structure of the proof summarized in Figure 1.

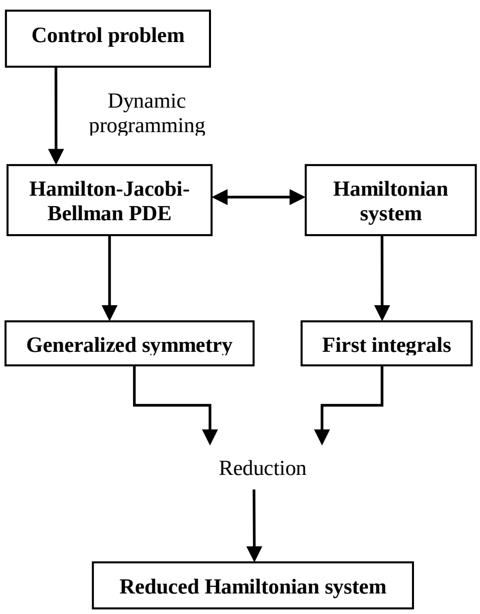

Figure 1 – Procedure of the proposed methodology

Solving a control problem by the proposed method

We consider the Hamilton-Jacobi-Bellman equation of optimal control problem (17) in $\Re^2\times(0,T)$, as the $u_t+\min_{a\in A}=\{(t\,\partial u/\partial x_1, t\,\partial u/\partial x_2).Du+(t(x_1x_2))^2\}=0$, for $A=[1,T]$ and arbitrary $T>1$. Then we have

$$u_t+\frac{1}{2}(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2})+(x_1-x_2)^2=0\cdot \qquad (22)$$

Step 1 (***Hamiltonian system).*** Let $M=\Re^4$ with canonical Poisson bracket, then of equation (21) applied to (22), the corresponded Hamiltonian function is of the form

$$H(p_1,p_2,x_1,x_2)=\frac{1}{2}(p_1^2+p_2^2)+(x_1-x_2)^2\cdot \qquad (23)$$

By (15) the corresponding Hamiltonian system is

$$\frac{dx_1}{dt}=p_1,\ \frac{dx_2}{dt}=p_2,\ \frac{dp_1}{dt}=-2(x_1-x_2),\ \frac{dp_2}{dt}=2(x_1-x_2). \qquad (24)$$

Step 2 (symmetry groups or first integrals). From (23) we find that the system admits an obvious translational invariance $\vec{v}=\partial_{x_1}+\partial_{x_2}$; the corresponding first integral is p_1+p_2.

Step 3 (reduction). According to Theorem 3, we introduce new coordinates $p=p_1+p_2, x=x_1, y=p_1, r=x_1-x_2$ which straighten out $\vec{v}=\partial_x$. In these variables, the Hamiltonian function is

$$H(p,y,r)=y^2-py+\frac{1}{2}p^2+r^2, \qquad (25)$$

and $\{F,H\}=\frac{\partial F}{\partial x}\frac{\partial H}{\partial y}+\frac{\partial F}{\partial r}\frac{\partial H}{\partial y}+\frac{\partial F}{\partial x}\frac{\partial H}{\partial p}-\frac{\partial F}{\partial y}\frac{\partial H}{\partial x}-\frac{\partial F}{\partial y}\frac{\partial H}{\partial r}-\frac{\partial F}{\partial p}\frac{\partial H}{\partial x}$.

Further, the Hamiltonian system is splitted into

$$\frac{dp}{dt}=-\frac{\partial H}{\partial x}=0,\ \frac{dx}{dt}=\frac{\partial H}{\partial p}+\frac{\partial H}{\partial y}=y,$$
$$\frac{dy}{dt}=-\frac{\partial H}{\partial x}-\frac{\partial H}{\partial r}=-2r,\ \frac{dr}{dt}=\frac{\partial H}{\partial y}=2y-p. \qquad (26)$$

Step 4 (solving of reduction system). The solution to the first pair, $p=c_1, x=\int y(t)dt+c_2$ (c_1,c_2 constant) can be determined from the solutions to the second pair (26). These form a reduced Hamiltonian system relative to the reduced Poisson bracket $\{\tilde{F},\tilde{H}\}=\tilde{F}_r\tilde{H}_y-\tilde{F}_y\tilde{H}_r$ for functions of y and r, with the Hamiltonian (25) obtained by fixing $p=c_1$.

For the explicit integration of (26) and, then, the exact solution of the original system (24), we use again the proof of Theorem 3; by setting $H(y,r)=\omega+(1/4)p^2$, we find $y^2-py+(1/2)p^2+V(r)=\omega+(1/4)p^2$ or $y=(1/2)p\pm\sqrt{\omega^2-r^2}$. In this way, we recover the solution just by integrating $dr/dt=2y-p\pm 2\sqrt{\omega-r^2}$ and we reach to exact solution

$$\pm\int dr/2\sqrt{\omega\text{-}r^2}=\int dt \text{ or } y=(1/2)p\pm\sqrt{(t\text{-}c)^2\text{-}r^2}\ .$$

Application of the proposed methodology to coordinate supplier development

Problem description and formulation

We consider the problem of coordinating supplier development programs in a two-echelon supply chain which is presented in Proch et al. [2]. The supply chain comprises of a single supplier and a single manufacturer in which the manufacturer assembles components from the supplier and sells the final product to the market. The goal is to find the optimal decision of supplier development investment.
A centralized decision-making process is assumed and the supply chain is considered as an integrated system in which all parameters, including the optimal amount of effort invested in supplier development, are simultaneously chosen. This decision-making process ensures system efficiency and opts for the optimum level of supplier development, i.e. maximizes the total profit of the supply chain. The variables and parameters for this model are summarized in Table 1.

Table 1 – Description of parameters and variables

Parameters/ Variables	Description
a	Prohibitive price (e.g. maximum willingness to pay)
b	Price elasticity of the commodity
c_M	Manufacturer's unit production costs
c_{SD}	Supply costs per unit charged by the supplier
c_0	Supplier's unit production cost at the begininig of the contract period
$x(t)$	The mesurement of the efforts invested in the supplier development
m	The supplier learning rate
$c_S(x)$	Supplier production cost
$c_S(x)=c_0x^m$	
$m=\dfrac{\ln(1\theta)}{\ln\chi}$;	
$\theta\in[0,1],\chi>1$	
r	The supplier fixed profit margin
$u(t)$	The effort at time t
$\omega(t)$	Capicity limit of $u(t)$ (resource availability in terms of time, man power or budget)

The profit function $J^{SC}:L^1([0,T),\Re)\to\Re$ of the set of measurable functions and the model of efforts invested in the supplier development is defined by the following problem

$$J^{SC}:=\int_0^T\frac{(a\text{-}c_M\text{-}c_0x^m(t))^2\text{-}r^2}{4b}\text{-}c_{SD}u(t)dt, \tag{27}$$
$$\text{subject to}\ \dot{x}=u;\ \ u:[0,T)\to[0,\omega),\ \ x(0)=x_0=1.$$

The centralized collaboration strategy should be determined such that the accumulated profit function (27) is maximized. By using the maximum principle (3) applied to the optimal control problem (27) with the Hamiltonian function

$$H(t,x,u,\lambda)=\frac{(a\text{-}c_M\text{-}c_0x^m(t))^2\text{-}r^2}{4b}\text{-}c_{SD}u(t)+\lambda(t)u(t), \tag{28}$$

the switching time t^* can be obtained by sulution $\partial H/\partial u(x^*,u^*(t),\lambda(t))=\text{-}c_{SD}+\lambda(t)=0$. Then, as examined in [2], t^* can be evaluated by numerical analysis from the equation

$$\frac{mc_0(1+\omega t^*)^{m1}(a\text{-}c_M\text{-}c_0(1+\omega t^*)^m)}{2b}(t^*\text{-}T)=c_{SD}\ . \tag{29}$$

More details of the above fromulation are given in [2].

Application of the proposed methodology

Optimal problem (27) is re the most known problems of the supplier-manufacturer relationship which result to an equation with different parameters for switching time (equation (29)) and then optimal control function which can be evaluated just by numerical analysis. In fact we have only one equation with different parameters (equation (29)). On the other hand, the case where the Hamiltonian H is linear in control u is of special interest. For one thing, it is an especially simple situation to handle when H plots against u as either a positively sloped or a negatively sloped straight line, since the optimal control is then always to be found at a boundary of u. The only task is to determine which boundary. More importantly, this case serves to highlight how a thorny situation in the calculus of variations has now become easily manageable in optimal control theory.
But, this simple approach apparently results in the elimination of some equations of the Hamiltonian system in supplier manufacturer coordination. For example because an accurate determination of the capacity limit $\omega=\omega(t)$ of $u(t)$ from the problem is not critical to our discussion, an accurate determination of it is exogenously assessed to be feasible to the problem.
However in our new approach, we consider all of functions and parameters in the system with their actual rule. Thus we can insert more variables in models of supply chain management. This can be modeled by considering some variables as a multiple functions with their rate of charges and then the Hamiltonian function as a function of all these unknown and their derivatives. In this way, we confront with possibly nonlinear optimal control problems which result to the systems of fully Hamiltonian equations with equations as equal as variables. Then, by Theorem 3 these complicated systems can be reduced Hamiltonian systems with exact solutions. The supply chain corresponded to

these models may be modeled for the most exact supplier-manufacturer relationship.

Solution method

From equation (27), we can rewrite the corresponding Hamiltonian function (28) as the

$H=H(\lambda,x,u,d)=d(p(d(t)-c_M-c_{SC})-c_{SD}u(t)+\lambda(t)u(t),$ (30)

with the production quantity $d(t)=a-c_M-c_{SC}/2b$ and the price distribution $p(d)=p(d(t))=a-bd=a+c_M+c_{SC}$; $c_{SC}=r+c_0x^m$. Then, we have the Hamiltonian system

$$\frac{\partial H}{\partial x}=\frac{-2mc_0x^{m-1}(t)(a-c_M-c_0x^m(t))}{4b}=-\frac{d\lambda}{dt},\ \frac{\partial H}{\partial \lambda}=u=\frac{dx}{dt}=\dot{x},$$

$$\frac{\partial H}{\partial d}=\dot{d}(a-bd-c_M-c_{SC})-bd=-\frac{du}{dt},\ \frac{\partial H}{\partial u}=-c_{SD}+\lambda=\dot{d},$$

which can be written as

$$\lambda(t)=\frac{mc_0x^{m-1}(t)(a-c_M-c_0x^m(t))}{2b}, \quad (31)$$

$$\dot{d}(t)=c_{SD}+\lambda(t), \quad (32)$$

$$\dot{u}(t)=\dot{d}(-a+bd+c_M+c_{SC})+bd\cdot \quad (33)$$

From equation (31) we have

$$\lambda(t)=\lambda(t^*)-\int_t^{t^*}\frac{mc_0x^{m-1}(s)(a-c_M-c_0x^m(s))}{2b}ds$$
$$=\lambda(t^*)-\frac{mc_0}{2b}(a-c_M)(I_{m-1}(t^*)-I_{m-1}(t)) \quad (34)$$
$$+\frac{mc_0^2}{2b}(I_{2m-1}(t^*)-I_{2m-1}(t)),$$

where $I_m(s)=\int_0^s x^m(k)dk$. Also, from equation (32)

$$d(t)=d(t^*)-\int_t^{t^*}(\lambda(s)-c_{SD})ds=d(t^*)-(t-t^*)c_{SD}-\int_t^{t^*}\lambda(s)ds\cdot$$

(35)

Then, substituting this to equation (33) results in

$$u(t)=u(t^*)+\frac{mc_0}{4b}(-a+c_M+r)(I_{m-1}(t^*)-I_{m-1}(t))$$
$$+\frac{mc_0^2}{4b}(I_{2m-1}(t^*)-I_{2m-1}(t)) \quad (36)$$
$$-\frac{1}{2}(a-c_M-r)(t^*-t)+\frac{c_0}{2}(I_m(t^*)-I_m(t))\cdot$$

Finally, for $x(t)=1+\omega t$, $t\in[0,t^*]$, as before of equation (34) we conclude

$$\lambda(t)=\lambda(t^*)-\frac{c_0(a-c_M)}{2b\omega}((1+\omega t^*)^m-(1+\omega t)^m)$$
$$+\frac{c_0{}^2}{4b\omega}((1+\omega t^*)^{2m}-(1+\omega t)^{2m}), \quad (37)$$

also, of (35)

$$d(t)=d(t^*)+\frac{c_0(a-c_M)}{2b\omega}(1+\omega t^*)^m(t^*-t)$$
$$-\frac{c_0(a-c_M)}{2b(m+1)\omega^2}((1+\omega t^*)^{m1}-(1+\omega t)^{m1})$$
$$-\frac{c_0{}^2}{4b\omega}((1+\omega t^*)^{2m}(t^*-t) \quad (38)$$
$$+\frac{c_0{}^2}{4b(2m+1)\omega^2}((1+\omega t^*)^{2m+1}-(1+\omega t)^{2m+1}),$$

and finally (36) result in

$$u(t)=u(t^*)+\frac{c_0}{4b\omega}(-a+c_M+r)((1+\omega t^*)^m-(1+\omega t)^m)$$
$$+\frac{mc_0{}^2}{8(m+1)\omega b}((1+\omega t^*)^{2m+2}-(1+\omega t)^{2m+2}) \quad (39)$$
$$-\frac{1}{2}(a-c_M-r)(t^*-t)+\frac{c_0}{2\omega(m+1)}((1+\omega t^*)^{m+1}-(1+\omega t)^{m+1})\cdot$$

On the other hand, $(-c_{SD}+p)\partial H/\partial p=u\,\partial H/\partial u$. Then, its first integral is $u=\pm\sqrt{2p^2-2c_{SD}}$ and (30) reduced to $H(p,x,d)=d(p(d(t))-c_M-c_{SC})\pm\sqrt{2p^2-2c_{SD}}(-c_{SD}+p)$.

Numerical example

In order to further illustrate applicability and superiority of the proposed methodology, a numerical example is presented using the data given in Proch et al. [2]. The parameter values are represented in Table 2.

Table 2 - Parameter values for numerical analysis (adopted from Proch et al. [2])

T	a	b	c_M	c_0	r	c_{SD}	ω	m
60	200	0.01	70	100	15	100000	1	-0.1

For the numerical analysis of the problem using the given parameter values, of equation (37) we obtain

$$\lambda(t^*)=\lambda(T)-\frac{c_0(a-c_M)}{2b\omega}((1+\omega T)^m-(1+\omega t^*)^m)$$
$$+\frac{c_0}{4b\omega}((1+\omega T)^{2m}-(1+\omega t^*)^{2m}),$$

since $\lambda(t^*)=c_{SD}$, then

$$100000=0-\frac{100(200-70)}{0.02}((1+60)^{-0.1}(1+t^*)^{-0.1})$$
$$+\frac{100}{0.04}(1+60)^{-0.2}-(1+t^*)^{-0.2}),$$

which results $t^*=9.844$. By substituting this in equation (37) we have

$$\lambda(t)=-25655-0650000\,(1+t)^{-0.1}-250000\,(1+t)^{-0.2},$$

and since $d(t^*)=(a-c_M-(rc_0x^{*m}))/2b=-19348.65$ then, from equation (38) we conclude

$$d(t)=-19348.65+512146.73(9.844-t)-722222.22(8.54-(1+t)^{0.9}$$
$$-155203.71(9.844-t)+312500(6.73-(1+t)^{0.8})\cdot$$

Also, from equation (39) we obtain

$$u(t)=u(t^{*})-287500(0.78-(1+t)^{-0.1})+13888.88(73-(1+t)^{1.8})$$
$$-57.5(9.88-t)+55.55(8.54-(1+t)^{0.9})\cdot$$

The approximate value of t^{*} which obtained numerically in Proch et al. [2] is 9.212. The advantage of our approach is that the exact value of the switching time which calculated analytically in our paper is 9.844 and it is certainly better result for our maximization control problem. Furthermore, the equations of $\lambda(t)$, $d(t)$ and $u(t)$ are obtained; while, in the original problem, an accurate determination of these variables has been exogenously assessed to be feasible or they should be approximately determined.

Conclusions

In this paper, a novel methodology was presented to find the exact optimal solution of the general control problem by developing a reformulation based on differential and Poisson geometry. For this purpose, we applied geometric notions about symmetric groups and first integrals to reduce the order of the Hamiltonian system. The proposed method was applied to supply chain coordination problem with the objective of finding the optimal decision of supplier development investment. We calculated the exact optimal solution and the optimum switching time for corresponding coordination problem of a supplier-manufacturer supply chain.

The main advantage of the proposed methodology is that it outperforms the numerical estimation approach since it provides the analytic solution of the problem and, thus, it yields better results than those obtained through numerical estimation. The proposed methodology can be successfully applied to the control problems in other optimization fields.

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