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Robustifying against Event-driven and Attribute-driven Uncertainties

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Abstract

Over five decades have passed since the first wave of robust optimization studies conducted by Soyster [1] *and Falk* [2]. *It is outstanding that real-life applications of robust optimization are still swept aside; there is much more potential for investigating the exact nature of uncertainties to have intelligent robust models. For this purpose, in this study, we investigate a more refined description of the uncertain events including (1) Event-driven and (2) Attribute-driven. Instead of model-based calibration of robustness, we analyze the structural properties of uncertain events to obtain a more refined description of the uncertainty polytopes. Hence, we introduce tractable robust models with a decent degree of conservatism and aversion from over-protection caused by the classic cardinality-restricted uncertainty polytopes.*

Keywords:
Robust optimization, Convex optimization, Uncertainty sets, Uncertainty events

Introduction

Robust optimization is a tractable alternative to stochastic programming particularly suited for the problems in which parameters are unknown and their respective distributions are uncertain. In many real-world situations, a precise stochastic description of the uncertain events may not be available. With less structured information, such as bounds of an uncertain parameter, one might describe the existing uncertainties by dedicating a set in which all realizations should lie, i.e. "uncertainty set". The goal is to guarantee the feasibility of the underlying constraints for any possible realizations, while optimizing an objective defending against the worst possible consequence.

The original form of robust optimization, introduced by Soyster [1] and Falk [2], was generally concerned with linear programming problems with inexact technological coefficients. Their proposed robust optimization was too conservative and subjected to driving the worst case for each uncertain parameter since the considered uncertainty was limited to the column-wise structure. Numerous works significantly generalized and extended the earlier platform into other classes of convex optimization problems beyond linear programming, e.g. conic and semi-definite programming (for example see [3] and [4]). The other works paved the way for a more complex description of the uncertainty polytopes, e.g. intersections of ellipsoidal uncertainty sets, budgeted uncertainty sets, etc. (for example see [5] and [6]). The key idea behind an uncertainty set is based on three components: nominal values of uncertain parameters, perturbation values, and uncertainty generating mechanism.

The rest of the paper is mainly focused on the uncertainty set proposed by Bertsimas and Sim [6], the so-called "cardinality-restricted uncertainty set".

$$S = \left\{ \xi \in R^n : \xi_i = \bar{\xi}_i + \delta_i \hat{\xi}_i,\ 0 \le \delta_i \le 1,\ \sum_i \delta_i \le \Gamma \right\} \tag{1}$$

The goal of this paper is to investigate more refined versions of uncertainty events. Hence, it enables us to avoid the over-protection issue caused by the classic cardinality-restricted uncertainty set. The generating polytope of the classic version benefits from convexity, especially integrality of its convex hull, which makes it more tractable. However, it is independent of decision variables of the refined model and type of uncertainty event. In this study, we present less conservative uncertainty sets which guarantee an improved protection level of the classic version. We discuss the types of events which generate the combinatorial structure of uncertainty sets, i.e. “combinatorial uncertainty set”. We also address the tractability issue of some problems caused by proposed uncertainty sets.

The rest of this paper is organized as follows. Section “Structural Properties of Uncertain Events” attempts to investigate the structural properties generated by the two common types of uncertainty sets. Section “Event & Attribute-driven Robustness” applies discussed uncertainty sets on two classic problems: robust knapsack problem and robust portfolio selection problem. The last section concludes the study with a summary and future directions.

Structural Properties of Uncertain Events

In this section, we briefly discuss the structural properties generated by the two common types of uncertainty sets; we call them “attribute-driven uncertainty set” (data-driven uncertainty set) and “event-driven uncertainty set” (combinatorial uncertainty set).

Attribute-driven uncertainty sets use the perturbed values of uncertain parameters as direct inputs to the mathematical model of the robust counterpart. They connect the decision-maker’s risk preferences with the “budget of uncertainty” and the controlling mechanism of uncertainty. For example, consider the portfolio selection problem in which an investor chooses the proportion of capital to be invested in each of N assets such that the desired wealth is achieved. The objective is to determine the fraction of invested asset i so as to maximize the total portfolio return. An underlying assumption of Markowitz’s model is that precise estimates of return of asset i, μ_i, and risk of asset i, σ_i, have been obtained. However, asset returns are uncertain. Hence, we can interpret risk of return i as the perturbed value of return of asset i and incorporate it in the uncertainty set.

$$S=\left\{\tilde{\mu}\in R^n : \tilde{\mu}_i=\bar{\mu}_i\pm\delta_i\sigma_i\,,\ 0\le\delta_i\le 1,\ \sum_i\delta_i\le\Gamma\right\} \tag{2}$$

In this example, the risks act as “internal” function of asset returns, also, any value of variable δ_i enforces asset i to gain risk value i.

On the other hand, the event-driven uncertainty set triggers uncertainty in parameters when a specific event is occurred in the system and generates discrete uncertainties.

$$U=\left\{\delta\in\{0,1\}^{|J|},\ \sum_{j\in J}\delta_j\le\Gamma\right\} \tag{3}$$

Uncertainty set U is non-convex and variable δ has the role to generate combination of events. The disruption systems are great examples of this type of uncertainty sets. An et al. [7] used a combinatorial uncertainty set to formulate a p-median facility location problem prone to disruptions. Moreover, uncertainty set U can be modified in order to encompass data of uncertain parameters. However, incorporating an event-driven uncertainty set into a mathematical model could make it computationally intractable. There exists two type of strategy, to the best of the authors, to solve the transformed model. First methods are mainly based on iterative algorithms. Zeng [8] proposed a column-and-constraint generation algorithm to solve two-stage robust optimization problems in which the first stage defined by a combinatorial uncertainty set. They also solve the model with existing benders-style cutting plane methods. The second type of methods, originally presented in this paper, transform event-driven uncertainty polytopes into attribute-driven ones. They are heuristically performed and differ from one problem to another. This method facilitates the transformation of original model into a tractable robust counterpart. We discuss this method in the next section.

Event & Attribute-driven Robustness

Robust Event-driven Knapsack Problem

Consider a knapsack problem given parameters c_i as the cost and w_i as the weight of item i.

$$\begin{aligned}&\text{maximize} && \sum_{i\in N}c_i x_i\\ &\text{subject to} && \sum_{i\in N}\tilde{w}_i x_i\le b\\ &x_i\in\square\,, && i\in N\end{aligned} \tag{4}$$

It is assumed that the weights w_i are uncertain, independently distributed and follow distributions in $[\bar{w}_i, \bar{w}_i+\hat{w}_i]$. The objective value vector c is not subject to data uncertainty. The goal is to maximize the total utility of $|N|$ items. The item should be selected and loaded on a cargo with strict weight restrictions. Suppose item i with weight $\bar{w}_i$ has auxiliary component i with weight $\hat{w}_i$. If uncertainty event occurs, $\delta_i=1$, the weight of item i should be increased by $\hat{w}_i$. We define the uncertainty set as follows.

$$U^{\Gamma}=\left\{w\in R^{|N|} : w_i=\bar{w}_i+\delta_i\hat{w}_i\,,\ \delta_i\in\{0,1\}\,,\ \sum_i\delta_i\le\Gamma\right\} \tag{5}$$

Then the original robust problem with respect to worst-case outcome becomes:

$$\begin{aligned}&\text{maximize} && \sum_{i\in N}c_i x_i\\ &\text{subject to} && \sum_{i\in N}\bar{w}_i x_i+\max_{U^{\Gamma}}\left\{\sum_{j\in U^{\Gamma}}\hat{w}_j x_j^*\right\}\le b\\ &x_i\in\square\,, && i\in N\end{aligned} \tag{6}$$

In order to solve the robust part of the problem (inner maximization) we heuristically define a second problem that seeks the worst-case outcome when the combination of events occurs.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j\in N} \hat{w}_j \delta_j x_j^* \\
\text{subject to} \quad & \sum_{j\in N} \delta_j \le \Gamma \\
& \delta_j \in \{0,1\}, \quad \forall j \in N
\end{aligned} \tag{7}
$$

In order to reach a closed form of the heuristic problem (7), we need to take the dual form of it. At first glance, the dualization technique may not be applicable since the problem is not a convex linear programming (variable δ_i is defined over integer domain). The following property addresses the integrality issue caused by the integer variable (we call this problem as "Ξ").

Property 1. *The convex hull of Ξ is an integral polyhedron***.**
Proof.
See Appendix.

■

Now we take the dual form of relaxed Ξ and incorporate it into the original problem. Relaxed Ξ is feasible and bounded for all Γ∈[0,|N|]. Also, the dual of relaxed Ξ is feasible and bounded. By strong duality, their objective values coincide.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i\in N} c_i x_i \\
\text{subject to} \quad & \sum_{i\in N} \bar{w}_i x_i + \tau\Gamma + \sum_{i} p_i \le b \\
& p_i + \tau \ge \hat{w}_i x_i \quad , \quad \forall i \in N \\
& \tau, p_i \ge 0, \quad \forall i \in N \\
& x_i \in \square \ , \quad \forall i \in N
\end{aligned} \tag{8}
$$

where τ and p_i are the dual variables of the relaxed Ξ.

Prioritized Budgeted Uncertainty Set

In practical cases, a limited "budget pool" is dedicated to both the item selection process and the robustness cost. In the previous example, the user only defines the budget of robustness; whereas the whole budget must be simultaneously addressed. Consider the previous example, the cargo (knapsack) problem. Suppose the cargo carries two set of items: medical items, Π, and commercial products, Π́. One cannot consider the underlying uncertainty of commercial products when necessary products are not loaded. On the other hand, to avoid the over-protection caused by U^{Γ}, which is independent of x, Poss [9] introduced a novel model of uncertainty polytope. Instead of considering budget of uncertainty Γ, a multifunction of x, $\gamma(x)$, was considered. We made some modifications to incorporate the prioritized budget into the uncertainty set. The objective still seeks to maximize the total utility of items. Given Φ as total budget for the robustness and set Π as the set of prioritized items,

$$
\Phi = \alpha \sum_{i\in\Pi} \mathbf{sign}(x_i^*)
$$

where α is the robustness importance. Then, the variable budget uncertainty set with the event-driven approach is proposed as follows.

$$
U^{\Phi} = \left\{ \tilde{w} \in R^{|N|} : \tilde{w}_i = \bar{w}_i + \delta_i \hat{w}_i, \delta_i \in \{0,1\}, \sum_{i\in N} \delta_i \le \Phi \right\} \tag{9}
$$

According to Property 1 and the worst-case criterion, the robust counterpart of this problem can be formulated as follows:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i\in N} c_i x_i \\
\text{subject to} \quad & \sum_{i\in N} \bar{w}_i x_i + \alpha\tau \sum_{i\in\Pi} \mathbf{sign}(x_i) + \sum_{i\in N} p_i \le b \\
& p_i + \tau \ge \hat{w}_i x_i \quad , \quad \forall i \in N \\
& \tau, p_i \ge 0, \quad \forall i \in N \\
& x_i \in \square \ , \quad \forall i \in N
\end{aligned} \tag{10}
$$

where τ and p_i are the dual variables. Note that the first constraint contains a non-linear expression. We make the following modifications and insertions in order to have a tractable linear form of the proposed counterpart.

$$
\begin{gathered}
y_i = \mathbf{sign}(x_i), \quad \forall i \in \Pi \\
x_i \le \left[\frac{b}{\bar{w}_i} \right] y_i ; \quad x_i \ge y_i , \quad \forall i \in \Pi \\
\sum_{i\in\Pi} \lambda_i = \tau \sum_{i\in\Pi} y_i , \\
\lambda_i \ge -\mathbf{M}(1-y_i) + \tau \ , \quad \forall i \in \Pi \\
\lambda_i \ge 0, \ y_i \in \{0,1\}, \quad \forall i \in \Pi
\end{gathered}
$$

The linear robust counterpart is presented as follows.

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i\in N} c_i x_i \\
\text{subject to} \quad & \sum_{i\in N} w_i x_i + \alpha \sum_{\forall i\in\Pi} \lambda_i + \sum_{i\in N} p_i \le b \\
& p_i + \tau \ge \hat{w}_i x_i \quad , \quad \forall i \in N \\
& x_i \le \left[\frac{b}{\bar{w}_i} \right] y_i , \quad \forall i \in \Pi \\
& x_i \ge y_i , \quad \forall i \in \Pi \\
& \lambda_i \ge -\mathbf{M}(1-y_i) + \tau \ , \quad \forall i \in \Pi \\
& \tau, p_i \ge 0, \quad \forall i \in N \\
& x_i \in \square \ , \quad \forall i \in N \\
& \lambda_i \ge 0 \ , \quad \forall i \in \Pi \\
& y_i \in \{0,1\} \ , \quad \forall i \in \Pi
\end{aligned} \tag{11}
$$

Property 2. *Constraints set* (9) *do not impose any additional restriction on the minimal value of* τ*, so that we can choose* **M** *equal to* $\mathbf{max}_{i\in N}(\hat{w}_i)$.

$$
\lambda_i \ge -\mathbf{max}_{i\in N}(\hat{w}_i)(1-y_i) + \tau \ , \quad \forall i \in \Pi
$$

Property 3. *If* α=1 *and* |Π|=|*N*|=Γ, then $\mathbf{opt}(U^{\Gamma}) = \mathbf{opt}(U^{\Phi})$.

Risk Compensatory Uncertainty Set

Bertsimas and Sim [6] reformulated a maximum expected return of a portfolio model as a linear robust optimization problem considering the classic cardinality-restricted uncertainty set.

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in N} \bar{r}_i x_i - \tau\Gamma - \sum_{i\in N} p_i \\ \text{subject to} \quad & \sum_{i\in N} x_i = 1, \\ & \tau + p_i \ge \hat{r}_i x_i \;, \quad \forall i \in N \\ & \tau, p_i \ge 0, \quad \forall i \in N \\ & x_i \in [0,1]\,, \quad \forall i \in N \end{aligned} \tag{12}$$

where C is the total capital, y_i is the portion of investment in asset i ($x_i=y_i/C$). The nominal value and the investment risk of return of asset i are $\bar{r}_i$ and $\hat{r}_i$. τ and p_i are the dual variables. Given N as the set of all assets. A less conservative approach (to avoid over-protection) is to compensate the robustness cost (the risk of obtaining lower profit) of "risky" assets, Ψ, by purchasing "safe" assets, Ψ'. The safe assets can be either a bank savings account or a government bond [10]. Hence, the return value of a subset of assets, Ψ', are prone to be perturbed within the predefined uncertainty interval (note that $\bar{r}_i > \bar{r}_j$ and $\hat{r}_i > \hat{r}_j$, $\forall i \in \Psi$, $\forall j \in \Psi'$). Given Φ as the budget of uncertainty,

$$\Phi = \alpha \sum_{i\in\Psi'} x_i^*$$

where α is the robustness importance. The uncertainty set is defined as follows.

$$U^{\Phi} = \left\{ \tilde{r} \in R^{|\Psi|} : \tilde{r}_i = \bar{r}_i \pm \delta_i \hat{r}_i, \delta_i \in [0,1], \sum_{i\in\Psi} \delta_i \le \Phi \right\} \tag{13}$$

Again, note that risk of asset i is an internal property of return of asset i (attribute-driven uncertainty); hence, we defined δ_i over a continuous domain. According to worst-case approach, the problem is formally stated as follows.

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in N} \bar{r}_i x_i - \mathbf{max}_{U^{\Phi}} \left\{ \sum_{j\in\Psi} \tilde{r}_j x_j^* \right\} \\ \text{subject to} \quad & \sum_{i\in N} x_i \le 1, \\ & x_i \in [0,1], \quad i \in N \end{aligned} \tag{14}$$

The inner maximization function of problem (14) can be translated into a single max-LP problem (15).

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in\Psi} \hat{r}_i \delta_i x_i^* \\ \text{subject to} \quad & \sum_{i\in\Psi} \delta_i \le \alpha \sum_{j\in\Psi'} x_j^* \;, \\ & \delta_i \in [0,1], \quad \forall i \in \Psi \end{aligned} \tag{15}$$

By dualization of problem (15) and strong duality theorem, the robust counterpart is obtained as (16).

$$\begin{aligned} \text{maximize} \quad & \sum_{i\in N} \bar{r}_i x_i - \tau(\alpha \sum_{i\in\Psi'} x_i) - \sum_{i\in\Psi} p_i \\ \text{subject to} \quad & \sum_{i\in N} x_i \le 1\,, \\ & p_i + \tau \ge \hat{r}_i x_i \;, \quad \forall i \in \Psi \\ & \tau, p_i \ge 0, \quad \forall i \in \Psi \\ & x_i \in [0,1], \quad \forall i \in N \end{aligned} \tag{16}$$

Property 4. $\tau \le \mathbf{max}_{i\in\Psi}(\hat{r_i})$.

Proof.

Consider deciding to invest the total capital on risky asset $k\in\Psi$ with the highest return, $\bar{r}_k = \mathbf{max}_{i\in\Psi}(\bar{r}_i)$, and also the highest risk, $\hat{r}_k = \mathbf{max}_{i\in\Psi}(\hat{r}_i)$. Hence, $x_k=1$ and $p_k+\tau \ge \hat{r}_k$. According to the objective function, variables p_k and τ should take the minimum possible value. Variable p_k does not operate in any other constraint and takes value zero; therefore, the minimum possible value for τ is $\hat{r}_k$. In other words, variable τ cannot exceed from $\mathbf{max}_{i\in\Psi}(\hat{r}_i)$.

■

The objective of problem (16) is not linear due to quadratic term $\tau(\alpha\Sigma_{i\in\Psi'} x_i)$. We reformulate this model based on McCormick convex envelope relaxation [11] and three following premises.

$$\begin{aligned} \lambda_i = \tau y_i \;, \quad & \forall i \in \Psi' \\ 0 \le x_i \le 1\,, \quad & \forall i \in \Psi' \\ 0 \le \tau \le \max_{i\in\Psi}(\hat{r}_i)\,. & \end{aligned}$$

Note that the efficiency of relaxation heavily depends on the tightness of the bounds of quadratic variables. The linear model of the robust counterpart is presented as follows.

$$\begin{aligned} & \text{maximize} \quad \sum_{i\in N} r_i x_i - \alpha \sum_{i\in\Psi'} \lambda_i - \sum_{i\in\Psi} p_i \\ & \text{subject to} \\ & \sum_{i\in N} x_i \le 1\,, \\ & p_i + \tau \ge \hat{r}_i x_i \;, \quad \forall i \in \Psi, \\ & \lambda_i \le \max_{j\in\Psi}(\hat{r}_j)\, x_i + \tau - \max_{j\in\Psi}(\hat{r}_j)\,, \quad \forall i \in \Psi', \\ & \lambda_i \le \max_{j\in\Psi}(\hat{r}_j)\,, \quad \forall i \in \Psi', \\ & \lambda_i \le \tau\,, \quad \forall i \in \Psi', \\ & \tau \le \max_{j\in\Psi}(\hat{r}_j)\,, \\ & \lambda_i = 0\,, \quad \forall i \in \Psi, \\ & \tau, p_i \ge 0, \quad \forall i \in \Psi, \\ & \lambda_i \ge 0, \quad \forall i \in \Psi', \\ & x_i \in [0,1], \quad \forall i \in N. \end{aligned} \tag{17}$$

Conclusion

In this study, we investigate the structural properties of uncertain events to have intelligent robust models with refined uncertainty polytopes. We also illustrated the approach by applying them on two classic robust optimization problems. There are cases when polyhedral theorems and affine policies does not operate for conversion of event-driven approaches to attribute-driven ones; hence, we recommend the adjustable robust models and we leave it as a future direction.

Appendix: Proof of Property 1.

This is equivalent to proving the technological coefficient of Ξ in the LP-relaxed form is totally uni-modular (TU). Now, we make use of the following proposition proved in [12]. Let $\boldsymbol{Q}$ be a matrix in {0, 1, -1} with no more than two nonzero elements in each column. Then $\boldsymbol{Q}$ is TU if and only if the rows of $\boldsymbol{Q}$ can be partitioned into two subsets $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$ such that if a column contains two nonzero elements and both nonzero elements have the same sign, then one is in a row contained in $\boldsymbol{Q_1}$ and the other is in a row contained in $\boldsymbol{Q_2}$.

In the LP-relaxed form, constraint set $\delta_i \leq 1$ should also be added to Ξ. The technological coefficients of Ξ are characterized as matrix $\boldsymbol{A}^{\Xi}$.

$$\boldsymbol{A}^{\Xi}_{(|N|+1)\times|N|} = \begin{pmatrix} \boldsymbol{e}_{1\times|N|} \\ \hline \boldsymbol{I}_{|N|\times|N|} \end{pmatrix}$$

$\boldsymbol{e}$ is an all-ones vector and $\boldsymbol{I}$ is a unit matrix. $\boldsymbol{Q}=\boldsymbol{A}^{\Xi}$ has components of 0 and +1 with no more than two nonzero entries in each column. $\boldsymbol{A}^{\Xi}$ can be partitioned into two subsets $\boldsymbol{Q_1}$ and $\boldsymbol{Q_2}$. Consider $\boldsymbol{Q_1}=\boldsymbol{e}$ and $\boldsymbol{Q_2}=\boldsymbol{I}$ (one of +1 is in a row contained in $\boldsymbol{Q_1}$ and the other is in a row contained in $\boldsymbol{Q_2}$). Under this condition, according to the proposition proved in [12], $\boldsymbol{A}^{\Xi}$ is TU. Then the relaxed Ξ is integral for $\Gamma \in \mathbb{Z}$.

■

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