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Estimation for the parameters of competing risks model with middle censored exponential data

K. Ahmadi*, M. Rezaei and F. Yousefzadeh
University of Birjand, K.Ahmadi@birjand.ac.ir
University of Birjand, Mjrezaei@birjand.ac.ir
University of Birjand, Fyousefzadeh@birjand.ac.ir

Abstract

In this paper, we consider some problems of estimation based on middle censored competing risks data. It is assumed that the lifetime distribution of the latent failure times are independent and exponential-distributed with the different parameters and also censoring mechanism is independent and non-informative.The maximum likelihood estimators of the unknown parameters are obtained. Based on gamma priors, the Lindely's approximation and Gibbs sampling methods are applied to obtain the Bayesian estimates of the unknown parameters under squared error loss function. Finally, a simulation study is given by Monte-Carlo simulations to evaluate the performances of the different methods.

Keywords: Competing risks, Middle censoring, Point estimation.

Mathematics Subject Classification (2010): 62N01, 62N02.

1 Introduction

In many life-testing studies, it is common that for the failure of the experimental unit there exist more than one cause of failure. The causes of failure are defined as risk factors before the failure occurs. Since these risk factors compete in some sense for the failure of the experimental unit, it is well known as the competing risks in the statistical literature. Some references in the field of the competing risks include [1] and [2]. In addition to competing risks, censoring is considered in many life test studies. Censoring is very common in life tests and it occurs for different reasons such as save to time, money and etc. The various categories of censoring are right, left and interval censoring. The middle censoring is a general concept of censoring which is introduced by [4]. In middle censoring the exact failure times are known only for a portion of the units under study and other failure times are unobservable and fall within random intervals. In this paper, we consider the competing risks model under middle censoring. Suppose $n \in \mathbb{N}$ identical units are put on a lifetime experiment and T_i denotes the lifetime of the ith unit then $T_i = min\{X_{i1}, X_{i2}, \ldots, X_{is}\}$, where X_{ij} is the latent failure time of the ith unit under the jth cause of failure, $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, s$. It is assumed that the latent failure times $X_{i1}, X_{i2}, \ldots, X_{is}$ are statistically independent and not observable, only T_i and C_i are observable where $C_i = j$ if failure is due to cause j. Moreover, it is assumed that X_{ij}

*Corresponding author and speaker

follows the exponential distribution with the probability density function(PDF) and cumulative distribution function(CDF) as $f_j(t;\theta_j) = \theta_j e^{-\theta_j t}$ and $F_j(t;\theta_j) = 1 - e^{\theta_j t}$ respectively, where $t, \theta_j > 0$ and θ_j is unknown parameter. The joint PDF and CDF of (X_i, C_i) are given by

$$f(t,j;\boldsymbol{\theta}) = \theta_j e^{-\theta_* t}, \text{ and } F(t,j;\boldsymbol{\theta}) = \frac{\theta_j}{\theta_*}\left(1 - e^{-\theta_* t}\right) \tag{1.1}$$

respectively, where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_s)$ and $\theta_* = \sum_{j=1}^{s} \theta_j$.

Let us consider the competing risks data with middle censoring scheme. Suppose $n \in \mathbb{N}$ identical units are put on a lifetime experiment. For the ith unit, there is a random censoring interval $[L_i, R_i]$, which is independent of the lifetime T_i. Also, for the ith unit, T_i is observable only if $T_i \notin [L_i, R_i]$, otherwise it is not observable. In other words, for the ith unit, we have

$$(Y_i, C_i, \delta_i) = \begin{cases} (T_i, C_i, 1) & T_i \notin [L_i, R_i] \\ ([L_i, R_i], C_i, 0) & T_i \in [L_i, R_i]. \end{cases}$$

Moreover, we assume that $(L_1, Z_1), (L_2, Z_2), \ldots, (L_n, Z_n)$ are i.i.d. where L_i and $Z_i = R_i - L_i$ are independent random variables and they have exponential distributions with the means α and β respectively. It is also assumed that α and β do not dependent on $\boldsymbol{\theta}$ and L_i and Z_i are independent of T_i.

2 Maximum likelihood estimators

Without loss of generality, after re-ordering the data, we assume that the first n_1 and rest n_2 are the uncensored and censored observations respectively. Hence, our observed data is:

$$\{(T_1, C_1, 1), \ldots, (T_{n_1}, C_{n_1}, 1), ([L_{n_1+1}, R_{n_1+1}], C_{n_1+1}, 0), \ldots, ([L_{n_1+n_2}, R_{n_1+n_2}], C_{n_1+n_2}, 0)\}$$

where $n_1 + n_2 = n$. Based on the observed data, the likelihood function can be written as

$$\begin{aligned} L(\boldsymbol{\theta}) &\propto \prod_{j=1}^{s}\prod_{i=1}^{n_1}\{f(t_i, j;\boldsymbol{\theta})\}^{I(C_i=j)} \prod_{j=1}^{s}\prod_{i=n_1+1}^{n_1+n_2}\{F(r_i, j;\boldsymbol{\theta}) - F(l_i, j;\boldsymbol{\theta})\}^{I(C_i=j)} \\ &\propto (\frac{1}{\theta_*})^{n-n_1}(\prod_{j=1}^{s}\theta_j^{m_j})e^{-\theta_*(\sum_{i=1}^{n_1} t_i + \sum_{i=1}^{n_1+n_2} l_i)} \prod_{i=n_1+1}^{n_1+n_2}\left(1 - e^{-\theta_* z_i}\right), \end{aligned} \tag{2.1}$$

where $z_i = r_i - l_i$,

$$I(C_i = j) = \begin{cases} 1 & \text{if } C_i = j \\ 0 & o.w., \end{cases}$$

and $m_j = \sum_{i=1}^{n} I(C_i = j)$ is total number of failures due to cause j. By setting the derivative of the log-likelihood function with respective to θ_j to zero, the likelihood equation is as

$$\frac{\partial \ln L(\boldsymbol{\theta})}{\partial \theta_j} = \frac{m_j}{\theta_j} - \frac{n_2}{\theta_*} - \sum_{i=1}^{n_1} t_i - \sum_{i=n_1+1}^{n_1+n_2} \ln l_i + \sum_{i=n_1+1}^{n_1+n_2} \frac{z_i e^{-\theta_* z_i}}{1 - e^{-\theta_* z_i}} = 0. \tag{2.2}$$

Based on the likelihood equations, we determine that the MLE of θ_j, say $\hat{\theta}_j$, must satisfy $\hat{\theta}_j = \frac{m_j}{n_1+n_2}\hat{\theta}_*$, $j = 1, 2, \ldots, s$, where $\hat{\theta}_*$ is the MLE of θ_*. Substituting back $\hat{\theta}_j$ in Eq.(2.2), $\hat{\theta}_*$ can be obtained from the equation

$K(\hat{\theta}_*) = 0$,

$$K(\hat{\theta}_*) = \frac{n_1}{\hat{\theta}_*} - \sum_{i=1}^{n_1} t_i - \sum_{i=n_1+1}^{n_1+n_2} l_i + \sum_{i=n_1+1}^{n_1+n_2} \frac{z_i e^{-\hat{\theta}_* z_i}}{1 - e^{-\hat{\theta}_* z_i}}. \tag{2.3}$$

Since $\lim_{\hat{\theta}_* \to 0} K(\hat{\theta}_*) = +\infty, \lim_{\hat{\theta}_* \to +\infty} K(\hat{\theta}_*) < 0$ and $\frac{\partial K(\hat{\theta}_*)}{\partial \hat{\theta}_*} < 0$, thus the equation $K(\hat{\theta}_*) = 0$ has only one root and the MLE of θ_j is unique for $j = 1, 2, \ldots, s$.

3 Bayesian point estimators

Now, we deal with the problem of Bayesian estimating the unknown parameter θ_j. Our prior knowledge about the true value of the unknown parameter θ_j is expressed via a gamma distribution with the PDF

$$\pi_j(\theta_j) = \frac{b_j^{a_j}}{\Gamma(a_j)} \theta_j^{a_j - 1} e^{-b_j \theta_j}, \quad a_j, b_j > 0,\ j = 1, 2, \ldots, s,$$

and it will be denoted by $Gamma(a_j, b_j)$. Here all the hyperparameters a_j and b_j are assumed to be known. Also, it is assumed that $\theta_1, \theta_2, \ldots, \theta_s$ have independent prior distributions. Based on the prior distributions and Eq.(2.1) we obtain the posterior density function of $\boldsymbol{\theta}$ given the data as

$$\pi(\boldsymbol{\theta}|data) = Cons. \prod_{j=1}^{s} \{\theta_j^{m_j + a_j - 1} e^{-\theta_j (b_j + \sum_{i=1}^{n_1} t_i + \sum_{i=n_1+1}^{n_1+n_2} l_i)}\} \prod_{i=n_1+1}^{n_1+n_2} \{\frac{1}{\theta_*}(1 - e^{-\theta_* z_i})\}. \tag{3.1}$$

It is obvious that the Bayesian estimator of θ_j under squared error loss function, $E(\theta_j|data)$, can not be obtained analytically.

3.1 Lindley's approximation

One of the most popular numerical techniques for the approximation of the ratio of integrals is Lindley's approximation method(see [3]). Using the Lindley's approximation, the Bayesian estimator of θ_j ("BL") under squared error loss function is as

$$\hat{\theta}_j^{SE} = \Big[\theta_j + \sum_{j'=1}^{s} (\frac{a_{j'} - 1}{\theta_{j'}} - b_{j'})\sigma_{jj'} + \frac{1}{2} \sum_{i'=1}^{s} \sum_{j'=1}^{s} \sum_{k=1}^{s} L_{i'j'k} \sigma_{i'j'} \sigma_{kj}\Big]_{\hat{\boldsymbol{\theta}}}, \tag{3.2}$$

where

$$L_{i'j'k} = \frac{\partial^3 \ln L(\boldsymbol{\theta})}{\partial \theta_{i'} \partial \theta_{j'} \partial \theta_k} = \begin{cases} \frac{2m_{j'}}{\theta_{j'}^3} - A & i' = j' = k \\ -A & o.w., \end{cases}$$

$$A = \frac{2n_2}{\theta_*^3} + \sum_{i=n_1+1}^{n_1+n_2} \frac{1 + e^{\theta_* z_i}}{(1 - e^{\theta_* z_i})^3} z_i^3 e^{\theta_* z_i},$$

$\hat{\boldsymbol{\theta}} = (\hat{\theta}_1, \hat{\theta}_2, \ldots, \hat{\theta}_s)$ and $\sigma_{i'j'} = (i', j')$-th elements of inverse matrix $[-L_{i'j'}]_{s \times s}, j = 1, 2, \ldots, s$. Also, it can be shown that

$$\sigma_{i'j'} = \begin{cases} \frac{1}{C}\Big(\prod_{\substack{j=1 \\ j \neq i'}}^{s} \frac{m_j}{\theta_j^2} + B \sum_{\substack{j_1 < j_2 < \ldots < j_{s-2} \\ j_1, j_2, \ldots, j_{s-2} \neq i'}} \frac{m_{j_1}}{\theta_{j_1}^2} \cdots \frac{m_{j_{s-2}}}{\theta_{j_{s-2}}^2} \Big) & i' = j' \\ -\frac{B}{C} \prod_{\substack{j=1 \\ j \neq i', j'}}^{s} \frac{m_j}{\theta_j^2} & i' \neq j', \end{cases}$$

where

$$B = \sum_{i=n_1+1}^{n_1+n_2} \frac{z_i^2 e^{\theta_* z_i}}{(e^{\theta_* z_i} - 1)^2} - \frac{n_2}{\theta_*^2},$$

$$C = \prod_{j=1}^{s} \frac{m_j}{\theta_j^2} + B \sum_{j_1<j_2<\ldots<j_{s-1}} \frac{m_{j_1}}{\theta_{j_1}^2} \cdots \frac{m_{j_{s-1}}}{\theta_{j_{s-1}}^2},$$

and for $s = 2$,

$$\sum_{\substack{j_1<j_2<\ldots<j_{s-2} \\ j_1,j_2,\ldots,j_{s-2}\neq i'}} \frac{m_{j_1}}{\theta_{j_1}^2} \cdots \frac{m_{j_{s-2}}}{\theta_{j_{s-2}}^2} = 1, \qquad \prod_{\substack{j=1 \\ j\neq i',j'}}^{s} \frac{m_j}{\theta_j^2} = 1 \tag{3.3}$$

$i', j', j_1, j_2, \ldots, j_{s-2}, j_{s-1} = 1, 2, \ldots, s$. The method requires that $\hat{\boldsymbol{\theta}}$ be unique which we proved its.

3.2 Gibbs sampling method

We propose to use the Gibbs sampling method to generate samples from the posterior density function (3.1). Since the conditional density of T, given $T \in [l, r]$ is

$$f_{T|T\in[l,r]}(t; \boldsymbol{\theta}) = \frac{\theta_* e^{-\theta_* t}}{e^{-\theta_* l} - e^{-\theta_* r}},\ l < t < r, \tag{3.4}$$

the posterior density in (3.1) can be rewritten as

$$\pi(\boldsymbol{\theta}|data) \propto \prod_{j=1}^{s} \{\theta_j^{m_j+a_j-1} e^{-\theta_j(b_j+\sum_{i=1}^{n_1} t_i + \sum_{i=n_1+1}^{n_1+n_2} t'_i)}\} \prod_{i=n_1+1}^{n_1+n_2} \left\{ \frac{1}{f_{T_i|T_i\in[l_i,r_i]}(t'_i; \boldsymbol{\theta})} \right\}. \tag{3.5}$$

Hence we use the below Algorithm to generate samples from the posterior density function of $\theta_1, \ldots, \theta_s$ from (3.5).

Algorithm 1: Step 1. Generate $\theta_j^{1,1}$ for $j = 1, 2, \ldots, s$ from $Gamma(m'_j + a_j, b_j + \sum_{i=1}^{n_1} t_i)$.

Step 2. Generate t'_{n_1+i} for $i = 1, 2, \ldots, n_2$ from $f_{T_{n_1+i}|T_{n_1+i}\in[l_{n_1+i}, r_{n_1+i}]}(.; \boldsymbol{\theta}^{1,1})$.

Step 3. Generate $\theta_j^{2,1}$ for $j = 1, 2, \ldots, s$ from $Gamma(m_j + a_j, b_j + \sum_{i=1}^{n_1} t_i + \sum_{i=n_1+1}^{n_1+n_2} t'_i)$.

Step 4. Go back to Step 2, and replace $\theta_j^{1,1}$ by $\theta_j^{2,1}$ for $j = 1, 2, \ldots, s$ and repeat Steps 2 and 3 for a large number say N times.

Therefore based on generated $\theta_j^{2,l'}, l' = 1, 2, \ldots, N$, the Bayesian estimate of θ_j ("BG") under squared error loss function can be obtained as $\frac{1}{N-b} \sum_{l'=b+1}^{N} \theta_j^{2,l'}$ where b is the burn-in sample.

4 Simulation study

In this section, we intend to investigate the performances of the different methods, as proposed in this paper. In this regard, some results based on Monte-Carlo simulations are presented. The simulation is carried out for $s = 2$, $(\theta_1, \theta_2) = (1, 0.5)$, $n = 20, 30, 50$, and for different censoring schemes. For censoring scheme we choose (α, β)=(0.5, 0.5)(scheme 1), (1.25, 0.5)(scheme 2), (1.5, 0.15)(scheme 3). The proportion of censoring (PC) under schemes 1-3 are 0.22, 0.37 and 0.52 respectively. All the Bayesian estimates are computed using

Table 1: The estimated biases and MSEs of the different estimators of θ_1 and θ_2.

				θ_1					θ_2				
					prior1		prior2			prior1		prior2	
(α, β)	PC	n		MLE	BL	BG	BL	BG	MLE	BL	BG	BL	BG
(0.5, 0.5)	0.22	20	Bias	0.0550	0.0573	0.0573	0.0151	0.0303	0.0287	0.0298	0.0298	0.0083	0.0160
Scheme 1			MSE	0.1008	0.1013	0.1013	0.0342	0.0505	0.0478	0.0480	0.0480	0.0179	0.0250
		30	Bias	0.0407	0.0422	0.0422	0.0234	0.0281	0.0200	0.0207	0.0207	0.0113	0.0137
			MSE	0.0625	0.0627	0.0627	0.0343	0.0396	0.0297	0.0297	0.0297	0.0169	0.0192
		50	Bias	0.0215	0.0223	0.0223	0.0162	0.0171	0.0097	0.0101	0.0101	0.0072	0.0076
			MSE	0.0351	0.0352	0.0352	0.0257	0.0269	0.0169	0.0170	0.0170	0.0126	0.0131
(1.25, 0.5)	0.37	20	Bias	0.0648	0.0698	0.0698	0.0144	0.0352	0.0374	0.0399	0.0399	0.0106	0.0214
Scheme 2			MSE	0.1122	0.1136	0.1136	0.0326	0.0527	0.0516	0.0522	0.0522	0.0176	0.0259
		30	Bias	0.0417	0.0450	0.0449	0.0219	0.0278	0.0211	0.0228	0.0228	0.0113	0.0142
			MSE	0.0700	0.0706	0.0706	0.0353	0.0424	0.0318	0.0320	0.0320	0.0172	0.0201
		50	Bias	0.0245	0.0265	0.0265	0.0187	0.0198	0.0124	0.0134	0.0134	0.0095	0.0101
			MSE	0.0385	0.0387	0.0387	0.0273	0.0288	0.0172	0.0173	0.0173	0.0126	0.0132
(1.5, 0.15)	0.52	20	Bias	0.0630	0.0697	0.0698	-0.0009	0.0295	0.0328	0.0362	0.0362	0.0001	0.0157
Scheme 3			MSE	0.1293	0.1301	0.1302	0.0281	0.0539	0.0565	0.0570	0.0570	0.0160	0.0261
		30	Bias	0.0433	0.0477	0.0478	0.0179	0.0269	0.0175	0.0197	0.0198	0.0058	0.0101
			MSE	0.0796	0.0800	0.0801	0.0337	0.0442	0.0336	0.0338	0.0339	0.0168	0.0203
		50	Bias	0.0235	0.0261	0.0261	0.0160	0.0178	0.0125	0.0138	0.0138	0.0086	0.0095
			MSE	0.0445	0.0446	0.0447	0.0288	0.0314	0.0194	0.0195	0.0195	0.0135	0.0143

none-informative and informative priors. For non-informative prior (prior1), $a_1 = a_2 = b_1 = b_2 = 0$ and for informative prior(prior 2), hyperparameters are taken as $a_1 = 4, b_1 = 4, a_2 = 2, b_2 = 4$. In order to obtain the Bayesian estimates using Gibbs sampling we set N = 10000 and b=1000. Under of these setting and based on M=10000 simulated middle censored samples the estimated biases and mean squared errors (MSEs)of "MLE", "BL" and "BG" are listed in Table1. From Table 1, It is observed that for a fixed PC, as the sample size n increases, the biases and MSEs decreases for "MLE" and Bayesian estimates based on prior 1. Also, for a fixed sample size, by increasing PC, MSEs of all estimates become a little bit large. As expected, the Bayesian estimates based on prior 2 are better than the corresponding Bayesian estimates based on prior 1. Also in the cases of prior 2, "BL" gives smaller biases and MSEs in comparison with "BG".

References

[1] K. Ahmadi, M. Rezaei and F. Yousefzadeh, Point predictors of the latent failure times of censored units in progressively Type-II censored competing risks data from the exponential distributions, *Journal of Statistical Computation and Simulation.* **86**(2016),1620–1634.

[2] M. J. Crowder, *Classical Competing Risks*, Chapman & Hall/CRC, New York, 2001.

[3] D. V. Lindley, Approximate bayesian methods, *Trabajos de Stadistca.* **31**(1980), 223–245.

[4] S. R. Jammalamadaka and V. Mangalam, Non-parametric estimation for middle censored data, *Journal of Nonparametric Statistics.* **15**(2003), 253–265.