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Generic continuity of KC-functions

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Abstract

We will show that every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous provided that X is Baire, Y a W-space and Z is regular. It follows that if Y is a Moore W-space and G is a Baire left topological group, then every KC-action $\pi : G \times Y \to Y$ is jointly continuous. Our results enable us to obtain conditions which imply that a semitopological group is automatically a paratopological.

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1. Introduction

There are many papers which deal with the problem of determining the points of continuity for a two variable function (see for example [2–4]). In particular, when the range of functions are not necessarily metric, any solution of the problem can be combined with group actions to emerge some strong results on joint continuity of group actions because the group actions spread the points of continuity around. One the most interesting result in this directs was obtained by Ellis [1] who proved that every separately continuous action $\pi : G \times X \to X$ is jointly continuous provided that G is a locally compact semitopological group and X is a locally compact space.

In this paper, we use a topological game argument to show that every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous, provided that X is a Baire space, Y is a W-space and Z is regular. In particular, when Z is a Moore space, it follows that for each $y \in Y$ the set of joint continuity of f is a dense G_δ subset of $X \times \{y\}$. We apply our results to prove that every KC-action $\pi : G \times Y \to Y$ is jointly continuous when Y is a Moore W-space and G is a Baire left topological group. In particular case, when $G = Y$, it follows that a Baire semitopological group G, is paratopological provided that G is a Moore W-space.

2. Results

In this section, we will introduce several topological games which will be used in the sequel. Each topological game is described by two types of rules; the playing

* speaker

rules, that determine how to play the game, and the winning rule which determines the winner. The winning rule differs from game to game and, actually, identifies the game.

Let (X,τ) be a topological space. The game $\mathcal{BM}(X)$ between two players α and β is done as follows. In step n, the player β picks a nonempty open set $U_n \subset V_{n-1}$, where U_{n-1} is the previous move of α-plaper, and α answers by selecting a nonempty open set $V_n \subset U_n$. The player α wins the play $(U_i, V_i)_{i\geq 1}$ if $(\bigcap_{n=1}^{\infty} V_n) \neq \emptyset$. Otherwise the player β is said to have won the play.
We say that the player α has a winning strategy for the game $\mathcal{BM}(X)$ if there exists a strategy s, such that α wins all plays provided that he/she acts according to the strategy s. In this case, we say that X is an α-favorable space, otherwise X is said to be an α-unfavorable space for this game. Similarly, winning strategy for the player β and β-favorablity are defined.

In 1976, Gruenhage defined a generalization of first countable spaces by means of the following topological game. Let Y be a topological space and $y_0 \in Y$. The topological game $\mathcal{G}(Y, y_0)$ is played by two players $\mathcal{O}$ and $\mathcal{P}$ as follows. In step n, $\mathcal{O}$-player selects an open set H_{n+1} with $y_0 \in H_{n+1}$ and then $\mathcal{P}$ answers by choosing a point $y_{n+1} \in H_{n+1}$. We say that $\mathcal{O}$ wins the game $g = (H_n, y_n)_{n\geq 1}$ if $y_n \to y_0$. We call $y \in Y$ a W-point (respectively w-point) in Y if $\mathcal{O}$ has (respectively $\mathcal{P}$ fails to have) a winning strategy in the game $\mathcal{G}(Y, y)$. A space Y in which each point of Y is a W-point (respectively w-point) is called a W-space (respectively w-space).

Let X, Y and Z be topological spaces, a function $\varphi : X \to Z$ is called *quasi-continuous at a point* $x \in X$ if for any neighborhood sets V of x and any neighborhood W of $\varphi(x)$ there exists a nonempty open set $G \subset V$ such that $\varphi(G) \subset W$. The function $\varphi : X \to Y$ is called *quasi-continuous* if it is quasi-continuous at each point of X. By a Kempisty continuous function (*KC-function* for short), we mean a function $f : X \times Y \to Z$ which is quasi-continuous in the first variable and continuous in the second variable.

A mapping $f : X\times Y \to Z$ is called *strongly quasi-continuous* at $(x, y) \in X\times Y$ if for each neighborhood W of $f(x, y)$ in Z and for each product of open sets $U \times V \subset X \times Y$ containing (x, y), there is a nonvoid open set $U_1 \subset U$ and a neighborhood $V_1 \subset V$ of y such that $f(U_1 \times V_1) \subset W$.

Theorem 2.1. *Let Y be a topological space and Z be a regular space. If either*
(1) X is a Baire space and the player $\mathcal{O}$ has a winning strategy in $\mathcal{G}(Y, y_0)$ or
(2) X is an α-favorable space and the player $\mathcal{P}$ does not have a winning strategy in $G(Y, y_0)$.
Then every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous on $X \times \{y_0\}$.

Sketch of the proof. If the result is not true, there is an open set W containing $z_0 = f(x_0, y_0)$ and there is some product of open sets $U\times V \subset X\times Y$ containing (x_0, y_0) such that for each open set $U' \subset U$ and each neighborhood $H' \subset H$ of y_0, there is some $(x', y') \in U' \times H'$ such that $f(x', y') \notin W$.

Since Z is regular, there is an open subset G with $f(x_0, y_0) \in G$ and $\overline{G} \subset W$. By quasi-continuity of $f(\cdot, y_0)$, there is a non-empty open subset $U' \subset U$ such that

$f(U' \times \{y_0\}) \subset G$. We define simultaneously a strategy s for β in $\mathcal{BM}(X)$ and a strategy t for $\mathcal{P}$ in $\mathcal{G}(Y, y_0)$ by induction as follows.

Let for $n \geq 1$, the partial plays $p_n = (U_i, V_i)_{i=1}^n$ in $\mathcal{BM}(X)$ and $g_n = (H_i, y_i)_{i=1}^n$ in $\mathcal{G}(Y, y_0)$ are specified. Since by our assumption $f(V_n \times H_n)$ is not contained in $\overline{G}$, there is some $(x_n, y_n) \in V_n \times H_n$ such that $f(x_n, y_n) \notin \overline{G}$. By quasi-continuity of $x \mapsto f(x, y_n)$, there is a non-empty open subset U_{n+1} of V_n such that $f(U_{n+1} \times \{y_n\}) \cap \overline{G} = \emptyset$. Define $s(U_1, V_1, \ldots, U_n, V_n) = U_{n+1}$ and $t(H_1, y_1, \ldots, H_n) = y_n$.

If (a) or (b) holds, there are a s-play $p = (U_n, V_n)$ and t-play $g = (H_n, y_n)$ which are won by α and $\mathcal{O}$ respectively. Let $x^* \in \bigcap_{n \geq 1} U_n$. There is an open neighborhood H of y_0 such that $f(x^*, y) \in G$ for all $y \in H$. Since $\mathcal{O}$ wins the play $g = (H_n, y_n)$, there is some $n_0 \in \mathbb{N}$ such that $y_n \in H$ for all $n \geq n_0$. Hence $f(x^*, y_0) \in G$. However, our construction shows that $f(x, y_n) \notin \overline{G}$ for all $x \in U_n$. This contradiction proves the Theorem. □

The following result follows immediately from Theorem 2.1.

Corollary 2.2. *Let Y be a topological space and Z be a regular space. If either*
(1) X is a Baire space and Y is a W-space or
(2) X is an α-favorable space and Y is a w-space.
Then every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous.

Let Z be a topological space $z \in Z$ and $\mathcal{U}$ be a collection of subsets of Z, then the *star of z with respect to $\mathcal{U}$* is defined by $st(z, \mathcal{U}) = \bigcup\{U \in \mathcal{U} : z \in U\}$. A sequence $\{\mathcal{G}_n\}$ of open covers of Z is said to be a *development* of Z if for each $z \in Z$, the set $\{st(z, \mathcal{G}_n) : n \in \mathbb{N}\}$ is a base at z. A *developable space* is a space which has a development. A *Moore* space is a regular developable space.

Corollary 2.3. *Let Z be a Moore space. If either*
(1) X is a Baire space and Y is a W-space or
(2) X is an α-favorable space and Y is a w-space.
Then for every KC-function $f : X \times Y \to Z$ and $y_0 \in Y$, there is a dense G_δ subset D_{y_0} of X such that f is jointly continuous at each point of $D_{y_0} \times \{y_0\}$.

A compact space Y is called *Corson compact* if for some κ, Y embeds in $\{\mathbf{x} \in \mathbb{R}^\kappa : x_\alpha = 0 \text{ for all but countably many } \alpha \in \kappa\}$.

Corollary 2.4. *Let X be a Baire space, Y be a Corson compact and Z be a regular space. Then every KC-function $f : X \times Y \to Z$ is strongly quasi-continuous. In particular, if Z is a Moore space, then f is jointly continuous on a dense subset of $X \times Y$.*

Proof. Since every Corson compact is a W-space, the result follows from Theorem 2.1 and Corollary 2.3. □

Let G be a group equipped with a topology. The group G is called left topological if for each $g \in G$, the left translation $h \in G \to gh \in G$ is continuous. By trivial change in the above definition, a right topological group can be defined. If G is both left and right topological, then G is called semitopological. A semitopological group is called paratopological if the product mapping is jointly continuous. If in addition the inverse function $x \mapsto x^{-1}$ is continuous, then G is said to be a topological group.

Let G be a left topological group and Y be a topological space. We say that G acts on X if there exists a function $\pi : G \times Y \to Y$ such that

$$\pi(gh, y) = \pi(g, \pi(h, y)) \quad (g, h \in G, y \in Y). \tag{1}$$

Theorem 2.5. *Let Y be a Moore space and G be a left topological group. If either*
(1) G is a Baire space and Y is a W-space or
(2) G is an α-favorable space and Y is a w-space.
Then every KC-action $\pi : G \times Y \to Y$ jointly continuous.

Proof. Let $(g_0, y_0) \in G \times Y$. By Corollary 2.3, there is a dense G_δ subset D_{y_0} of G such that π is jointly continuous at each point of $D_{y_0} \times \{y_0\}$. Let $\{g_\alpha\}$ and $\{y_\alpha\}$ converge to g and y_0 respectively and take some arbitrary point $h \in D_{y_0}$. Since π is continuous at (h, y_0) and

$$\lim_\alpha hg^{-1}g_\alpha = h, \quad \lim_\alpha y_\alpha = y_0,$$

we see that $\lim_\alpha \pi(hg^{-1}g_\alpha, y_\alpha) = \pi(h, y_0)$. Therefore by using (1), we have

$$\begin{aligned} \lim_\alpha \pi(g_\alpha, y_\alpha) &= \lim_\alpha \pi(gh^{-1}, \pi(hg^{-1}g_\alpha, y_\alpha)) \\ &= \pi(gh^{-1}, \pi(h, y_0)) = \pi(g, y_0). \end{aligned}$$

This proves our result. □

The following result which follows from Theorem 2.5, generalizes [4, Theorem 4].

Corollary 2.6. *Let G be a semitopological group which is also a Moore space. If G is a Baire W-space, then it is a paratopological group.*

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