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# On the continuous wave packet frames

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**Abstract**

Consider the wave packet group $G_\Theta = H \ltimes_\Theta (K \times \widehat{K})$ where $H$ and $K$ are locally compact groups, $K$ is also abelian and $\Theta : H \to Aut(K \times \widehat{K})$ is a continuous homomorphism. In this article, we extend the notation of Zak transform on $L^2(G_\Theta)$ and introduce a frame condition to generate wave packet frames on $G_\Theta$.

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## 1. Introduction

The Zak transform has been used in applications in physics and signal theory [6]. An approach to define the Zak transform on semidirect product groups of the form $G_\tau = H \ltimes_\tau K$ introduced in [3]. Many locally compact groups are non-abelian although they can be cosidered as semidirect product of locally compact groups. Dual of non-abelian locally compact groups is based on the dual group. But the dual of non abelian locally compact groups is considerably more intricate and consists of all classes of equivalence of its irreducible representations [2]. In this paper, by using a version of duality for semidirect product group, we extend some classical results from abelian case to non-abelian semidirect product groups which are compatible and useful in application. Throughout this article, we assume $G_\tau = H \ltimes_\tau K$ is the semi-direct product group of locally compact group $H$ and locally compact abelian group $K$. The mapping $h \mapsto \tau_h$ is a homomorphism of $H$ into the group of automorphisms of $K$ such that the mapping $(h, k) \longmapsto \tau_h(k)$ from $H \times K$ onto K is continuous. The group law is given by

$$(h, k).(h', k') = (hh', k\tau_h(k')), \quad ((h, k) \in G_\tau).$$

Then $G_\tau$ is a (not necessarily abelian) locally compact group. Moreover, the left Haar measure of $G_\tau$ is $dm_{G_\tau}(h, k) = \delta(h)dm_H(h)dm_K(k)$, where $m_H$ and $m_K$ are the left Haar measures of $H$ and $K$, respectively and the positive continuous homomorphism $\delta$ on $H$ is given by (15.29 of [5])

$$dm_K(k) = \delta(h)dm_K(\tau_h(k)). \tag{1}$$

* speaker

It is worthwhile to mention that the above $\tau$-dual action on $K$ induces such $\tau$-dual on $\widehat{K}$. More precisely, we can define homomorphism $\widehat{\tau} : H \to Aut(\widehat{K})$ via $h \mapsto \widehat{\tau}_h$, given by

$$\widehat{\tau}_h(\omega) := \omega_h = \omega \circ \tau_{h^{-1}} \tag{2}$$

for all $\omega \in \widehat{K}$, where $\omega_h(k) = \omega(\tau_{h^{-1}}(k))$ for all $k \in K$. Also, the continuity of the homomorphism $\widehat{\tau} : H \to Aut(\widehat{K})$ given in (2) guaranteed by Theorem 26.9 of [5]. Hence, the semi-direct product $G_{\widehat{\tau}} = H \times_{\widehat{\tau}} \widehat{K}$ is a locally compact group with the left Haar measure

$$dm_{G_{\widehat{\tau}}}(h, \omega) = \delta(h)^{-1} dm_H(h) dm_{\widehat{K}}(\omega). \tag{3}$$

Furthermore, the continuous action $(h, k) \mapsto \tau_h(k)$ induces a mapping $\Theta : H \to (K \times \widehat{K})$ given by $h \mapsto \Theta_h$ where

$$\Theta_h(k, \omega) = (\tau_h(k), \omega_h). \tag{4}$$

In [3], it is shown that $\Theta$ is a well-defined homomorphism and $(h, k, \omega) \mapsto \Theta_h(k, \omega)$ is continuous. Thus $\Theta$ induces the semi direct product group $G_\Theta = H \ltimes_\Theta (K \times \widehat{K})$ which is called the *wave packet group*. It is a locally compact group with the left Haar measure

$$dm_\Theta(h, k, \omega) = dm_H(h) dm_K(k) dm_{\widehat{K}}(\omega)$$

and the modular function

$$\Delta_{G_\Theta}(h, k, \omega) = \Delta_H(h),$$

for all $(h, k, \omega) \in G_\Theta$, for more details see Theorem 3.2 of [3]. For $(k, \omega) \in K \times \widehat{K}$and $f \in L^2(K \times \widehat{K})$ the translation operator $T_{(k,\omega)} : L^2(K \times \widehat{K}) \to L^2(K \times \widehat{K})$ is defined by

$$T_{(k,\omega)} f(x, \xi) = f(xk^{-1}, \xi\overline{\omega}).$$

Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a *frame* for $\mathcal{H}$ if there are constants $A, B > 0$ satisfying

$$A \parallel f \parallel^2 \leq \sum_{k=0}^{\infty} \mid \langle f, f_i \rangle \mid^2 \leq B \parallel f \parallel^2 .$$

If $\{f_i\}_{i=1}^{\infty}$ is a frame, the *frame operator* is defined by

$$S : \mathcal{H} \to \mathcal{H}, \quad Sf = \sum_{k=0}^{\infty} \langle f, f_i \rangle f_i.$$

The series converging unconditionally and $S$ is a bounded, invertible, and self-adjoint operator. This leads to the frame decomposition

$$f = S^{-1} S f = \sum_{i=1}^{\infty} \left\langle f, S^{-1} f_i \right\rangle f_i.$$

## 2. Main result

The Zak transform was first introduced and used in 1950 by Gelfand for a problem in differential equations. Weil defined this transform on arbitrary locally compact abelian group with respect to arbitrary closed subgroup. Subsequently the Zak transform was redicovered in quantum mechanic by Zak. Finally the continuous Zak transform for locally compact groups is established in [1].

Definition 2.1. *The Zak transform of $f \in C_c(G_\theta)$ is defined on $K \times \widehat{K}$ by*

$$Z_c f(h,\omega,x) = \int f(h,y,\gamma)\overline{\omega(y)\gamma(x)}dm_k(y)dm_{\widehat{K}}(\gamma).$$

*It is shown that $Z_c : C_c(G_\theta) \to C_c(\widehat{G_\theta})$ is an isometry in $L^2$-norms and so that it can be uniquely extended into the Zak transform $Z_c : L^2(G_\Theta) \to L^2(G_{\widehat{\Theta}})$.*

If $f = f_1 \otimes g_1$ that $f_1 \in L^2(H)$ and $g_1 \in L^2(K \times \widehat{K})$ then

$$f_1 \otimes g_1 \in L^2(H) \otimes L^2(K \times \widehat{K}) = L^2(H \times K \times \widehat{K}).$$

Also

$$\begin{aligned} \|Z_c f\|_2^2 &= \int_{H\times K\times\widehat{K}} \left| \int f_1(h) g_1(y,\gamma)\overline{\omega(y)\gamma(x)}dm_K(y)dm_{\widehat{K}}(\gamma)\right|^2 dm_H(h)dm_{\widehat{K}}(\omega)dm_K(x) \\ &= \int_H |f_1(h)|^2 dm_H(h) \int_{K\times\widehat{K}} \left| \int_{K\times\widehat{K}} g_1(y,\gamma)\overline{\omega(y)\gamma(x)}dm_K(y)dm_{\widehat{K}}(\gamma)\right|^2 dm_{\widehat{K}}(\omega)dm_K(x) \\ &= \|f_1\|_2^2 . \left\|\widehat{g_1}\right\|_2^2 . \end{aligned}$$

Therefore $Z_c f = f_1 \otimes \widehat{g_1}$ and

$$\begin{aligned} \|Z_c f\|_2 &= \|f_1\|_2 \|\widehat{g_1}\|_2 \\ &= \|f_1\|_2 \, \|g_1\|_2 \\ &= \|f_1 \otimes g_1\|_2 \\ &= \|f\|_2 . \end{aligned}$$

Let H be a locally compact group and K an locally compact abelian group with the dual group $\widehat{K}$ . For $h \in H$ and $f \in L^2(K \times \widehat{K})$ define the dilation of $f$ by h via

$$D_h f(k,\omega) = f(\tau_{h^{-1}}(k), \widehat{\tau}_{h^{-1}}(\omega)).$$

Theorem 2.2. *Let $G_\Theta = H \ltimes_\theta (K \times \widehat{K})$ and $\psi \in L^2(K \times \widehat{K})$. Then $\mathcal{F}(\psi) = \{T_{(k,\omega)} D_h \psi;\quad (h,k,\omega) \in G_\theta\}$ is a continuous frame with bounds $A, B$ if and only if $A \le \gamma_\psi \le B$ a.e. where $\gamma_\psi = \int_H |(Z_c D_h \psi)(\xi, y)|^2 dm_H(h)$.*

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