Submitted to the 5st Seminar on Harmonic Analysis and Applications
Organized by the Iranian Mathematical Society
January 18–19, 2017, Ferdowsi University of Mashhad, Iran

پنجمین سمینار
آنالیز هارمونیک
و کاربردها
5th
Seminar on
Harmonic Analysis
and Applications

Solving the wave equation with shearlet frames

M. Amin khah*, A. Askari Hemmat and R. Raisi Tousi

Abstract

In this paper, we apply parseval shearlet frames to solve the wave equation. To this end, using the Plancherel's theorem, we calculate the shearlet coefficients.

2010 *Mathematics subject classification:* Primary 42C15, Secondary 42C40.
Keywords and phrases: Shearlet system, wave equation, tight frame.

1. Introduction

In the context of seismic oil and gas exploration, the wave equation is the important factor to establish the link between the earth properties and the observed data at the surface. In the past, several traditional methods are proposed for solving the wave equation, including the finite difference method [1], the pseudospectral method [2], the finite element method. In recent years, wavelets are used to solve the wave equation. Compared with wavelets, it seams that shearlets are more efficient for solving the wave equation [5].

A *discrete shearlet system* associated with $\psi \in L^2(\mathbb{R}^2)$ is defined by

$$\{\psi_{j,k,m} = a_0^{-\frac{3}{4}j}\psi(S_k A_{a_0^{-j}} \cdot -m) : \; j,k \in \mathbb{Z}, m \in \mathbb{Z}^2\}, \; a_0 > 0, \tag{1}$$

where the *parabolic scaling matrices* A_{a_0} and the *shearing matrix* S_k are given by

$$A_{a_0} = \begin{bmatrix} a_0 & 0 \\ 0 & a_0^{\frac{1}{2}} \end{bmatrix}, \qquad S_k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}. \tag{2}$$

The *discrete shearlet transform* of $f \in L^2(\mathbb{R}^2)$ is the mapping defined by

$$f \mapsto \mathcal{SH}_\psi f(j,k,m) = \langle f, \psi_{j,k,m} \rangle, \quad (j,k,m) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}^2.$$

* speaker

2. Main result

We start with the construction of a tight frame of shearlets for $L^2(\mathbb{R}^2)$. Let ϕ be the scaling function of a Meyer wavelet. For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, let $\hat{\Phi}(\xi) = \hat{\Phi}(\xi_1, \xi_2) = \hat{\phi}(\xi_1)\hat{\phi}(\xi_2)$ and

$$W_1(\xi) = W_1(\xi_1, \xi_2) = \sqrt{\hat{\Phi}^2(2^{-2}\xi_1, 2^{-2}\xi_2) - \hat{\Phi}^2(\xi_1, \xi_2)}.$$

It follows that

$$\hat{\Phi}^2(\xi_1, \xi_2) - \sum_{j\geq 0} W_1^2(2^{-2j}\xi_1, 2^{-2j}\xi_2) = 1, \quad \text{for } (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Notice that $W_{1_j}^2 = W_1^2(2^{-2j}\cdot)$. In particular, the functions $W_{1_j}^2$, $j \geq 0$, produce a smooth tiling of the frequency plane into Cartesian coronae: $\sum_{j\geq 0} W_1^2(2^{-2j}\xi) = 1, \quad \text{for } \xi \in \mathbb{R}^2 \backslash [-\frac{1}{8}, \frac{1}{8}]^2 \subset \mathbb{R}^2$. Next, let $W_2 \in C^\infty(\mathbb{R})$ be chosen so that $\text{supp} W_2 \subset [1, 1]$ and

$$|W_2(\zeta - 1)|^2 + |W_2(\zeta)|^2 + |W_2(\zeta + 1)|^2 = 1, \quad \text{for } |\zeta| \leq 1.$$

In addition, we will assume that $W_2(0) = 1$ and that $W_2^{(n)}(0) = 0$ for all $n \geq 1$. Using this notation we state the following definition [3, 4].

Definition 2.1. *For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, the shearlet system for $L^2(\mathbb{R}^2)$ is defined as the countable collection of functions*

$$\{\psi_{j,l,k} : j \geq 0, -2^j \leq l \leq 2^j, k \in \mathbb{Z}^2\}, \tag{3}$$

where

$$\hat{\psi}_{j,l,k}(\xi) = |detA_4|^{-\frac{j}{2}} W_1(2^{-2j}\xi) W_2(\xi A_4^{-j} S_1^{-l}) e^{2\pi i \xi A_4^{-j} S_1^{-l} k},$$

and A_4, S_1 are given by (2).

Using the above observation it can be shown that the shearlet system (3) is a tight frame [3].

Theorem 2.2. *The shearlet system (3) is a tight frame for $L^2(\mathbb{R}^2)$.*

Because of the shearlet system (3) forms a tight frame, we can expand a function $f \in L^2(\mathbb{R}^2)$ as a series of shearlet

$$f = \sum_{j,l,k} \langle f, \psi_{j,l,k} \rangle \psi_{j,l,k}, \tag{4}$$

We denote the shearlet coefficients $\langle f, \psi_{j,l,k} \rangle$ by $C_{j,l,k}$. According to the Plancherel's theorem, we have

$$\begin{aligned} C_{j,l,k} &= \langle f, \psi_{j,l,k} \rangle = \langle \hat{f}, \hat{\psi}_{j,l,k} \rangle \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\xi) \overline{\hat{\psi}_{j,l,k}(\xi)} d\xi \\ &= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{f}(\xi) \overline{2^{-\frac{3}{2}j} W_1(2^{-2j}\xi) W_2(\xi A_4^{-j} S_1^{-l}) e^{2\pi i \xi A_4^{-j} S_1^{-l} k}} d\xi \end{aligned} \tag{5}$$

Now with the aid of shearlet coefficients we solve the wave equation. We consider the wave equation as follows

$$\frac{\partial^2 u(x,t)}{\partial t^2} = c^2 \Delta u(x,t),$$

$$u(x,t)|_{t=0} = u_1(x), \quad \frac{\partial u(x,t)}{\partial t}|_{t=0} = u_2(x),$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ and $\Delta u(x,t) = \frac{\partial^2 u(x,t)}{\partial x_1^2} + \frac{\partial^2 u(x,t)}{\partial x_2^2}$. The shearlet coefficients (5) for u are defined by

$$C_{j,l,k} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \hat{u}(\xi) \overline{\hat{\psi}_{j,l,k}(\xi)} d\xi,$$

$$C^{\Delta}_{j,l,k} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \widehat{\Delta u}(\xi) \overline{\hat{\psi}_{j,l,k}(\xi)} d\xi.$$

By the properties of Fourier transform for derivative, we have

$$\widehat{\Delta u}(\xi_1, \xi_2) = ((i\xi_1)^2 + (i\xi_2)^2)\hat{u}(\xi) = -|\xi|^2 \hat{u}(\xi). \tag{6}$$

Finally, setting (6) in (5) and applying a change of variables, we obtain the coefficient shearlet associated to u. substituting the coefficient shearlet in (4), we obtain the desirable result.

References

[1] R. M. Alford, K. R. Kelly and D. M. Boore, Accuracy of finite difference modeling of the acoustic wave equation, *Geophysics* **39** (1974) 834-842.

[2] J. Gazdag, Modeling of the acoustic wave equation with transform methods, *Geophysics* **46.6** (1981) 854-859.

[3] K. Guo and D. Labate, The construction of smooth Parseval frames of shearlets, *Mathematical Modelling of Natural Phenomena* **8.1** (2013) 82-105.

[4] S. Häuser and G. Steidl, Fast finite shearlet transform, *arXiv preprint arXiv:1202.1773* **–** (2012)

[5] G. Kutyniok and W. Q. Lim, Compactly supported shearlets are optimally sparse, *Journal of Approximation Theory* **163.11** (2011) 1564-1589.

M. Amin khah,
Department of Mathematics, Faculty of Sciences and new Technologies,
Graduate University of Advanced Technology,
Kerman, Iran.
e-mail: m.aminkhah@student.kgut.ac.ir

A. Askari Hemmat,
Department of Applied Mathematics, Faculty of Mathematics and Computer,
Shahid Bahonar Uninersity of Kerman,
Kerman, Iran
e-mail: askari@uk.ac.ir

R. Raisi Tousi,
Department of Pure Mathematics,
Ferdowsi University of Mashhad,
Mashhad, Iran
e-mail: raisi@um.ac.ir