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# Derivations on the Banach algebra $L^\infty(G)^*$ of a locally compact group

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**Abstract**

In this paper, we present some results concerning derivations and Jordan derivations on the noncommutative Banach algebra $L^\infty(G)^*$. We show that the range of any (skew-) centralizing derivation on $L^\infty(G)^*$ is embedded into $M(L^\infty, C(G))$, and the zero map is the only skew-commuting derivation on $L^\infty(G)^*$. We also prove that the subspace $Ann_r(L^\infty(G)^*)$ of $L^\infty(G)^*$ is invariant under Jordan derivations.

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## 1. Introduction

Let $G$ be a locally compact group with a fixed left Haar measure $\lambda$. The Banach space of complex-valued integrable functions with respect to $\lambda$ is denoted by $L^1(G)$. With the norm $||.||_1$ and with convolution

$$\phi * \psi(x) = \int_G \phi(y)\psi(y^{-1}x)\, d\lambda(y) \quad (x \in G)$$

as product, $L^1(G)$ becomes a Banach algebra. Let $L^\infty(G)$, the usual Lebesgue space as defined in [2] equipped with the essential supremum norm $||.||_\infty$. It is well-known that the dual of $L^\infty(G)$, represented by $L^\infty(G)^*$, is a Banach algebra with the first Arens product " $\cdot$ " defined by the formula

$$\langle m \cdot n, f\rangle = \langle m, nf\rangle,$$

where

$$\langle nf, \phi\rangle = \langle n, f\phi\rangle, \quad \text{and} \quad \langle f\phi, \psi\rangle = \langle f, \phi * \psi\rangle$$

for all $m, n \in L^\infty(G)^*$, $f \in L^\infty(G)$ and $\phi, \psi \in L^1(G)$; for example see [3]. Note that we may consider $\phi \in L^1(G)$ as a linear functional in $L^\infty(G)$ by the formula

$$\langle \phi, f\rangle = \int_G f(x)\phi(x)\, d\lambda(x)$$

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for all $f \in L^\infty(G)$. Let $\Lambda(G)$ denote the set of all weak$^*$-cluster points of an approximate identity in $L^1(G)$ bounded by one. It is easy to see that if $u \in \Lambda(G)$, then

$$m \cdot u = m \quad \text{and} \quad u \cdot \phi = \phi$$

for all $m \in L^\infty(G)^*$ and $\phi \in L^1(G)$. The right annihilator of $L^\infty(G)^*$ is denoted by $\text{Ann}_r(L^\infty(G)^*)$ and is the set of all $r \in L^\infty(G)^*$ such that $L^\infty(G)^* \cdot r = \{0\}$. It is well-known from [4] that $L^\infty(G)^*$ is the direct sum of the norm closed right ideal $u \cdot L^\infty(G)^*$ and $\text{Ann}_r(L^\infty(G)^*)$ for all $u \in \Lambda(G)$. Let

$$\begin{aligned} M(L^\infty, C(G)) = \{\Phi : \Phi &\text{ is a continuous linear map from } L^\infty(G) \text{ to } C(G) \quad \text{and} \\ &\Phi(\psi * f) = \psi * \Phi(f) \text{ for all } \psi \in L^1(G), f \in L^\infty(G)\} \end{aligned}$$

where $C(G)$ is the space of all complex-valued bounded continuous functions on $G$. One can show that

$$M(L^\infty, C(G)) = \cap_{u\in\Lambda(G)} u.L^\infty(G)^*;$$

see [4].

A derivation on $L^\infty(G)^*$ is a linear mapping $d$ on $L^\infty(G)^*$ satisfying

$$d(m \cdot n) = d(m) \cdot n + m \cdot d(n).$$

A derivation $d$ on $L^\infty(G)^*$ is called *centralizing* if

$$[d(m), m] \in Z(L^\infty(G)^*)$$

for all $m \in L^\infty(G)^*$. Also, derivation $d$ is called *skew-centralizing* if

$$d(m) \cdot m + m \cdot d(m) \in Z(L^\infty(G)^*),$$

for all $m \in L^\infty(G)^*$, where $Z(L^\infty(G)^*)$ is the center of $L^\infty(G)^*$, the set of all $m \in L^\infty(G)^*$ such that $m \cdot n = n \cdot m$ for all $n \in L^\infty(G)^*$ and

$$[m, n] := m \cdot n - n \cdot m$$

for all $m, n \in L^\infty(G)^*$.

Centralizing derivations have been studied by Posner [5]. He showed that the zero map is the only centralizing derivation on a noncommutative prime ring. In [1] Bresar proved that there is no nonzero additive mapping in a prime ring $R$ of characteristic different from 2 which is skew-commuting on $R$.

Let us remark that if $G$ is a nondiscrete group, then Hahn-Banach theorem shows that there is a nonzero element $r \in \text{Ann}_r(L^\infty(G)^*)$. Hence for every $u \in \Lambda(G)$, we have $u \cdot r = 0$ and $r \cdot u = r$ and

$$r \cdot L^\infty(G)^* \cdot r = \{0\}.$$

So $L^\infty(G)^*$ is a noncommutative Banach algebra. This also implies that $L^\infty(G)^*$ is not a semiprime ring. Hence we cannot apply the well-known results concerning derivations of commutative Banach algebras and derivations of prime rings to $L^\infty(G)^*$. In this paper we investigate the truth of these results for $L^\infty(G)^*$.

In this paper, we investigate (skew-) centralizing derivations on $L^\infty(G)^*$ and prove that the range of any (skew-) centralizing derivation on $L^\infty(G)^*$ is embedded into $M(L^\infty, C(G))$. We also prove that the zero map is the only skew-commuting derivation on $L^\infty(G)^*$. Finally, we show that the subspace $Ann_r(L^\infty(G)^*)$ of $L^\infty(G)^*$ is invariant under Jordan derivations.

## 2. Main Results

We commence this section with the main result of the paper.

Theorem 2.1. *Let $G$ be a locally compact group and $d$ be a derivation on $L^\infty(G)^*$ such that*

$$[[d(m), m], m] = 0$$

*for all $m \in L^\infty(G)^*$. Then the range of $d$ is embedded into $M(L^\infty, C(G))$.*

As an immediate consequence of Theorem 2.1 we have the following result.

Corollary 2.2. *Let $G$ be a locally compact group and $d$ be a centralizing derivation on $L^\infty(G)^*$. Then the range of $d$ is embedded into $M(L^\infty, C(G))$.*

Now, we prove a result concerning skew-centralizing derivations.

Theorem 2.3. *Let $G$ be a locally compact group and $d$ be a skew-centralizing derivation on $L^\infty(G)^*$. Then the range of $d$ is embedded into $M(L^\infty, C(G))$.*

A derivation $d$ on $L^\infty(G)^*$ is called *skew-commuting* if

$$d(m) \cdot m + m \cdot d(m) = 0,$$

for all $m \in L^\infty(G)^*$.

Theorem 2.4. *The zero map is the only skew-commuting derivation on $L^\infty(G)^*$.*

A linear mapping $d$ on $L^\infty(G)^*$ is said to be a *Jordan derivation* if

$$d(m^2) = d(m) \cdot m + m \cdot d(m)$$

for all $m \in L^\infty(G)^*$.

Theorem 2.5. *Let $G$ be a locally compact group and $d$ be a Jordan derivation on $L^\infty(G)^*$. Then the following statements hold.*

(a) *$d$ maps $Ann_r(L^\infty(G)^*)$ into $Ann_r(L^\infty(G)^*)$.*

(b) *$d(u \cdot m) = d(u) \cdot m + u \cdot d(m)$ for all $m \in L^\infty(G)^*$ and $u \in \Lambda(G)$. Moreover, $d$ maps $\Lambda(G)$ into $Ann_r(L^\infty(G)^*)$.*

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