

Duals of K-fusion frames

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Abstract

In this paper, we consider the notion of *K*-fusion frames in Hilbert spaces, which is a new generalization of fusion frames. Our main purpose is to reconstruct the elements from the range of the bounded operator K on a Hilbert space \mathcal{H} by using a family of closed subspaces in \mathcal{H} . For this end, we introduce the notion of duality for *K*-fusion frames and characterize duals of some *K*-fusion frames. Also, we survey the robustness of *K*-fusion frames under some perturbations.

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1. Introduction

Let \mathcal{H} be a separable Hilbert space and I be a countable index set, a sequence $F := \{f_i\}_{i \in I} \subseteq \mathcal{H}$ is called a *K*-frame for \mathcal{H} , if there exist constants A, B > 0 such that

$$A\|K^*f\|^2 \le \sum_{i\in I} |\langle f, f_i \rangle|^2 \le B\|f\|^2, \quad (f \in \mathcal{H}).$$

$$\tag{1}$$

K-frames [5] were introduced to reconstruct elements from the range of a bounded linear operator $K \in B(\mathcal{H})$. Clearly, if $K = I_{\mathcal{H}}$, then *F* is an ordinary frame and so *K*-frames arise as a generalization of the ordinary frames. Authors in [1] introduced the notion of duality for *K*-frames and presented some approaches for construction and characterization of *K*-duals.

Fusion frame theory is a natural generalization of frame theory in separable Hilbert spaces, which is introduced by P. Casazza and G. Kutyniok in [2]. Fusion frames are applied in signal and image processing, sampling theory, filter banks and a variety of applications that cannot be modeled by discrete frames. In the following, we review basic definitions and results of fusion frames.

Let $\{W_i\}_{i \in I}$ be a family of closed subspaces of \mathcal{H} and $\{\omega_i\}_{i \in I}$ a family of weights, i.e. $\omega_i > 0, i \in I$. Then $\{(W_i, \omega_i)\}_{i \in I}$ is called a *fusion frame* for \mathcal{H} if there exist the

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constants $0 < A \le B < \infty$ such that

$$A||f||^{2} \leq \sum_{i \in I} \omega_{i}^{2} ||\pi_{W_{i}}f||^{2} \leq B||f||^{2}, \qquad (f \in \mathcal{H}),$$
(2)

where π_{W_i} denotes the orthogonal projection from Hilbert space \mathcal{H} onto a closed subspace W_i .

The constants A and B are called the *fusion frame bounds*. If we only have the upper bound in (2) we call $\{(W_i, \omega_i)\}_{i \in I}$ a *Bessel fusion sequence*. For each sequence $\{W_i\}_{i \in I}$ of closed subspaces in \mathcal{H} , the space

$$\sum_{i\in I} \oplus W_i = \left\{ \{f_i\}_{i\in I} : f_i \in W_i, \sum_{i\in I} ||f_i||^2 < \infty \right\},\$$

with the inner product $\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle$ is a Hilbert space. For every Bessel fusion sequence $W := \{(W_i, \omega_i)\}_{i\in I}$ of \mathcal{H} , the synthesis operator $T_W : \sum_{i\in I} \oplus W_i \to \mathcal{H}$ is defined by $T_W(\{f_i\}_{i\in I}) = \sum_{i\in I} \omega_i f_i$, for each $\{f_i\}_{i\in I} \in \sum_{i\in I} \oplus W_i$. The analysis operator, is given by $T_W^*(f) = \{\omega_i \pi_{W_i}(f)\}_{i\in I}, f \in \mathcal{H}$ and the fusion frame operator $S_W : \mathcal{H} \to \mathcal{H}$ is defined by $S_W f = \sum_{i\in I} \omega_i^2 \pi_{W_i} f$, which is a bounded, invertible and positive operator on \mathcal{H} [2].

Throughout this paper, we suppose *I* is a countable index set and $I_{\mathcal{H}}$ is the identity operator on \mathcal{H} . For two Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 we denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the collection of all bounded linear operators between \mathcal{H}_1 and \mathcal{H}_2 , and we abbreviate $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. Also we denote the range of $K \in B(\mathcal{H})$ by R(K), and the orthogonal projection of \mathcal{H} onto a closed subspace $V \subseteq \mathcal{H}$ is denoted by π_V .

2. Main results

In this section, we introduce the notion of dual for *K*-fusion frames in Hilbert spaces. Then we, try to identify and characterize duals of some *K*-fusion frames.

Definition 2.1. Let $\{W_i\}_{i \in I}$ be a family of closed subspaces of \mathcal{H} and $\{\omega_i\}_{i \in I}$ be a family of weights, i.e. $\omega_i > 0$, $i \in I$. We call $W = \{(W_i, \omega_i)\}_{i \in I}$ a K-fusion frame for \mathcal{H} , if there exist positive constants $0 < A, B < \infty$ such that

$$A||K^*f||^2 \le \sum_{i \in I} \omega_i^2 ||\pi_{W_i}f||^2 \le B||f||^2, \qquad (f \in \mathcal{H}).$$
(3)

The constants *A* and *B* in (3) are called lower and upper bounds of *W*, respectively. We call *W* a minimal *K*-fusion frame, whenever $W_i \cap \overline{span}_{j \in I, j \neq i} W_j = \{0\}$. Obviously, a *K*-fusion frame is a Bessel fusion sequence and so the synthesis operator, the analysis operator and the frame operator of *W* are defined similar to fusion frames, however for a *K*-fusion frame, the synthesis operator is not onto and the frame operator is not invertible, in general.

Now, for a *K*-fusion frame $W = \{(W_i, \omega_i)\}_{i \in I}$ and a Bessel sequence $V = \{(V_i, \upsilon_i)\}_{i \in I}$, we define $\phi_{\upsilon w} : \sum_{i \in I} \oplus V_i \to \sum_{i \in I} \oplus W_i$ by

$$\phi_{vw}\{f_i\}_{i\in I} = \{\pi_{W_i}(S_W^{-1})^* K f_i\}_{i\in I}$$

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Clearly ϕ_{vw} is a bounded linear operator, and we will have the following definition, where is a generalization of duality introduced by Găvruţa in [4].

Definition 2.2. Let $W = \{(W_i, \omega_i)\}_{i \in I}$ be a K-fusion frame. A Bessel sequence $\{(V_i, \upsilon_i)\}_{i \in I}$ is called a K-dual of W if

$$Kf = T_W \phi_{vw} T_V^* f = \sum_{i \in I} \omega_i \upsilon_i \pi_{W_i} (S_W^{-1})^* K \pi_{V_i} f, \quad (f \in \mathcal{H}).$$

$$\tag{4}$$

If $\widetilde{W} := \{(K^*S_W^{-1}\pi_{S_W(R(K))}W_i, \omega_i)\}_{i \in I}$ is a Bessel sequence, then we can easily see that \widetilde{W} is a *K*-dual for *W*; it is called the canonical *K*-dual of *W*. However, in the next example we construct a *K*-fusion frame for which not only \widetilde{W} is not a Bessel fusion sequence, but also it has no *K*-dual.

Example 2.3. Let $\mathcal{H} = l^2$ with the standard orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Define

$$Ke_{i} = \begin{cases} \sum_{m=1}^{\infty} \frac{1}{m^{2}} e_{2m-1} & i = 1, \\ 0 & i = 2, \\ e_{8m} & i = m+2, \quad (m \in \mathbb{N}). \end{cases}$$

Then $K \in B(\mathcal{H})$, and if we consider the subspaces $W_2 = \overline{span}\{e_2 + e_4\}$, $W_4 = \overline{span}\{e_2 + e_4\}$, and $W_n = \overline{span}\{e_n\}$, for all $n \neq 2, 4$. Then $\{(W_n, 1)\}$ is a K-fusion frame. Moreover, a direct calculation shows that every sequence $V = \{(V_i, v_i)\}_{i=1}^{\infty}$ satisfies (4) is not a Bessel fusion sequence, i.e., W has no K-dual.

It is worth noticing that, when $\{(\pi_{S_W(R(K))}W_i, \omega_i)\}_{i \in I}$ is a Bessel fusion sequence, then \widetilde{W} is also a Bessel fusion sequence. In the next proposition we show that for some *K*-fusion frames many *K*-duals may be exist.

Proposition 2.4. Suppose that $K \in B(\mathcal{H})$ is a closed range operator and $W = \{(W_i, \omega_i)\}_{i \in I}$ is a K-fusion frame. If \widetilde{W} is a Bessel fusion sequence, then W has at least a K-dual different from the canonical K-dual.

The next result characterize all K-duals of minimal K-fusion frames.

Theorem 2.5. Let $W = \{(W_i, \omega_i)\}_{i \in I}$ be a minimal K-fusion frame for \mathcal{H} and \overline{W} be a Bessel fusion sequence. The a Bessel fusion sequence $V = \{(V_i, \omega_i)\}_{i \in I}$ is a K-dual of W if and only if $K^*S_W^{-1}\pi_{S_W(R(K))}W_i \subseteq V_i$, for all $i \in I$.

3. Perturbation of K-fusion frames

Stability of fusion frames have been considered in [3]. In this section, we study robustness of *K*-fusion frames under some perturbations.

Theorem 3.1. Let $W = \{(W_i, \omega_i)\}_{i \in I}$ be a K-fusion frame for \mathcal{H} with bounds A and B, respectively. Also, let $Z = \{(Z_i, z_i)\}_{i \in I}$ be a $(\lambda_1, \lambda_2, \varepsilon)$ -perturbation of W for some $0 < \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$, *i.e.*,

$$\|(\omega_i \pi_{W_i} - z_i \pi_{Z_i})f\| \le \lambda_1 \|\omega_i \pi_{W_i}f\| + \lambda_2 \|z_i \pi_{Z_i}\| + \varepsilon \omega_i \|K^*f\|,$$

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for all $i \in I$ and $f \in \mathcal{H}$. such that

$$\varepsilon < \frac{(1-\lambda_1)\sqrt{A}}{\|K\|(\sum_{i\in I}\omega_i^2)^{1/2}}$$

Then Z is a K-fusion frame for \mathcal{H} .

Finally, in the following theorem we show that under some small perturbations, *K*-duals of a *K*-fusion frame turn to the approximate *K*-dual for perturbed *K*-fusion frame.

Theorem 3.2. Let $W = \{(W_i, \omega_i)\}_{i \in I}$ be a K-fusion frame for \mathcal{H} with bounds A and B, respectively. Also, let $Z_i \subseteq \mathcal{H}$ be closed subspace of \mathcal{H} for all i. If $\varepsilon > 0$ such that

$$\|(T_W^*-T_Z^*)f\|<\varepsilon\|K^*f\|.$$

- (i) If $0 < \varepsilon < \sqrt{A}$, then $Z = \{(Z_i, \omega_i)\}_{i \in I}$ is a K-fusion frame with bounds $(\sqrt{A} \varepsilon)$ and $(\sqrt{B} + \varepsilon ||K||)$, respectively.
- (ii) If $\varepsilon > 0$ is sufficiently small, then every K-dual $V = \{(V_i, \upsilon_i)\}_{i \in I}$ of W satisfies $||K T_Z \phi_{\upsilon_Z} T_V^*|| < 1.$

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