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Duals of K-fusion frames

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Abstract

In this paper, we consider the notion of K-fusion frames in Hilbert spaces, which is a new generalization of fusion frames. Our main purpose is to reconstruct the elements from the range of the bounded operator K on a Hilbert space $\mathcal{H}$ by using a family of closed subspaces in $\mathcal{H}$. For this end, we introduce the notion of duality for K-fusion frames and characterize duals of some K-fusion frames. Also, we survey the robustness of K-fusion frames under some perturbations.

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1. Introduction

Let $\mathcal{H}$ be a separable Hilbert space and I be a countable index set, a sequence $F := \{f_i\}_{i\in I} \subseteq \mathcal{H}$ is called a K-frame for $\mathcal{H}$, if there exist constants $A, B > 0$ such that

$$A\|K^* f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{1}$$

K-frames [5] were introduced to reconstruct elements from the range of a bounded linear operator $K \in B(\mathcal{H})$. Clearly, if $K = I_{\mathcal{H}}$, then F is an ordinary frame and so K-frames arise as a generalization of the ordinary frames. Authors in [1] introduced the notion of duality for K-frames and presented some approaches for construction and characterization of K-duals.

Fusion frame theory is a natural generalization of frame theory in separable Hilbert spaces, which is introduced by P. Casazza and G. Kutyniok in [2]. Fusion frames are applied in signal and image processing, sampling theory, filter banks and a variety of applications that cannot be modeled by discrete frames. In the following, we review basic definitions and results of fusion frames.

Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ a family of weights, i.e. $\omega_i > 0$, $i \in I$. Then $\{(W_i, \omega_i)\}_{i\in I}$ is called a *fusion frame* for $\mathcal{H}$ if there exist the

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constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \omega_i^2 \|\pi_{W_i} f\|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}), \tag{2}$$

where π_{W_i} denotes the orthogonal projection from Hilbert space $\mathcal{H}$ onto a closed subspace W_i.

The constants A and B are called the *fusion frame bounds*. If we only have the upper bound in (2) we call $\{(W_i, \omega_i)\}_{i\in I}$ a *Bessel fusion sequence*. For each sequence $\{W_i\}_{i\in I}$ of closed subspaces in $\mathcal{H}$, the space

$$\sum_{i\in I} \oplus W_i = \left\{ \{f_i\}_{i\in I} : f_i \in W_i, \sum_{i\in I} \|f_i\|^2 < \infty \right\},$$

with the inner product $\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I} \rangle = \sum_{i\in I} \langle f_i, g_i \rangle$ is a Hilbert space. For every Bessel fusion sequence $W := \{(W_i, \omega_i)\}_{i\in I}$ of $\mathcal{H}$, the *synthesis operator* $T_W : \sum_{i\in I} \oplus W_i \to \mathcal{H}$ is defined by $T_W(\{f_i\}_{i\in I}) = \sum_{i\in I} \omega_i f_i$, for each $\{f_i\}_{i\in I} \in \sum_{i\in I} \oplus W_i$. The *analysis operator*, is given by $T_W^*(f) = \{\omega_i \pi_{W_i}(f)\}_{i\in I}$, $f \in \mathcal{H}$ and the *fusion frame operator* $S_W : \mathcal{H} \to \mathcal{H}$ is defined by $S_W f = \sum_{i\in I} \omega_i^2 \pi_{W_i} f$, which is a bounded, invertible and positive operator on $\mathcal{H}$ [2].

Throughout this paper, we suppose I is a countable index set and $I_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. For two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ we denote by $B(\mathcal{H}_1, \mathcal{H}_2)$ the collection of all bounded linear operators between $\mathcal{H}_1$ and $\mathcal{H}_2$, and we abbreviate $B(\mathcal{H}, \mathcal{H})$ by $B(\mathcal{H})$. Also we denote the range of $K \in B(\mathcal{H})$ by $R(K)$, and the orthogonal projection of $\mathcal{H}$ onto a closed subspace $V \subseteq \mathcal{H}$ is denoted by π_V.

2. Main results

In this section, we introduce the notion of dual for K-fusion frames in Hilbert spaces. Then we, try to identify and characterize duals of some K-fusion frames.

Definition 2.1. *Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ be a family of weights, i.e. $\omega_i > 0$, $i \in I$. We call $W = \{(W_i, \omega_i)\}_{i\in I}$ a K-fusion frame for $\mathcal{H}$, if there exist positive constants $0 < A, B < \infty$ such that*

$$A\|K^* f\|^2 \leq \sum_{i\in I} \omega_i^2 \|\pi_{W_i} f\|^2 \leq B\|f\|^2, \qquad (f \in \mathcal{H}). \tag{3}$$

The constants A and B in (3) are called lower and upper bounds of W, respectively. We call W a minimal K-fusion frame, whenever $W_i \cap \overline{span}_{j\in I, j\neq i} W_j = \{0\}$. Obviously, a K-fusion frame is a Bessel fusion sequence and so the synthesis operator, the analysis operator and the frame operator of W are defined similar to fusion frames, however for a K-fusion frame, the synthesis operator is not onto and the frame operator is not invertible, in general.

Now, for a K-fusion frame $W = \{(W_i, \omega_i)\}_{i\in I}$ and a Bessel sequence $V = \{(V_i, \upsilon_i)\}_{i\in I}$, we define $\phi_{vw} : \sum_{i\in I} \oplus V_i \to \sum_{i\in I} \oplus W_i$ by

$$\phi_{vw}\{f_i\}_{i\in I} = \{\pi_{W_i} (S_W^{-1})^* K f_i\}_{i\in I}.$$

Clearly ϕ_{vw} is a bounded linear operator, and we will have the following definition, where is a generalization of duality introduced by Găvruţa in [4].

Definition 2.2. *Let $W = \{(W_i, \omega_i)\}_{i\in I}$ be a K-fusion frame. A Bessel sequence $\{(V_i, \upsilon_i)\}_{i\in I}$ is called a K-dual of W if*

$$Kf = T_W \phi_{vw} T_V^* f = \sum_{i\in I} \omega_i \upsilon_i \pi_{W_i} (S_W^{-1})^* K \pi_{V_i} f, \quad (f \in \mathcal{H}). \tag{4}$$

If $\widetilde{W} := \{(K^* S_W^{-1} \pi_{S_W(R(K))} W_i, \omega_i)\}_{i\in I}$ is a Bessel sequence, then we can easily see that $\widetilde{W}$ is a K-dual for W; it is called the canonical K-dual of W. However, in the next example we construct a K-fusion frame for which not only $\widetilde{W}$ is not a Bessel fusion sequence, but also it has no K-dual.

Example 2.3. *Let $\mathcal{H} = l^2$ with the standard orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Define*

$$Ke_i = \begin{cases} \sum_{m=1}^{\infty} \frac{1}{m^2} e_{2m-1} & i = 1, \\ 0 & i = 2, \\ e_{8m} & i = m+2, \quad (m \in \mathbb{N}). \end{cases}$$

Then $K \in B(\mathcal{H})$, and if we consider the subspaces $W_2 = \overline{span}\{e_2 + e_4\}$, $W_4 = \overline{span}\{e_2 + e_4\}$, and $W_n = \overline{span}\{e_n\}$, for all $n \neq 2, 4$. Then $\{(W_n, 1)\}$ is a K-fusion frame. Moreover, a direct calculation shows that every sequence $V = \{(V_i, \upsilon_i)\}_{i=1}^{\infty}$ satisfies (4) is not a Bessel fusion sequence, i.e., W has no K-dual.

It is worth noticing that, when $\{(\pi_{S_W(R(K))} W_i, \omega_i)\}_{i\in I}$ is a Bessel fusion sequence, then $\widetilde{W}$ is also a Bessel fusion sequence. In the next proposition we show that for some K-fusion frames many K-duals may be exist.

Proposition 2.4. *Suppose that $K \in B(\mathcal{H})$ is a closed range operator and $W = \{(W_i, \omega_i)\}_{i\in I}$ is a K-fusion frame. If $\widetilde{W}$ is a Bessel fusion sequence, then W has at least a K-dual different from the canonical K-dual.*

The next result characterize all K-duals of minimal K-fusion frames.

Theorem 2.5. *Let $W = \{(W_i, \omega_i)\}_{i\in I}$ be a minimal K-fusion frame for $\mathcal{H}$ and $\widetilde{W}$ be a Bessel fusion sequence. Then a Bessel fusion sequence $V = \{(V_i, \omega_i)\}_{i\in I}$ is a K-dual of W if and only if $K^* S_W^{-1} \pi_{S_W(R(K))} W_i \subseteq V_i$, for all $i \in I$.*

3. Perturbation of K-fusion frames

Stability of fusion frames have been considered in [3]. In this section, we study robustness of K-fusion frames under some perturbations.

Theorem 3.1. *Let $W = \{(W_i, \omega_i)\}_{i\in I}$ be a K-fusion frame for $\mathcal{H}$ with bounds A and B, respectively. Also, let $Z = \{(Z_i, z_i)\}_{i\in I}$ be a $(\lambda_1, \lambda_2, \varepsilon)$-perturbation of W for some $0 < \lambda_1, \lambda_2 < 1$ and $\varepsilon > 0$, i.e.,*

$$\|(\omega_i \pi_{W_i} - z_i \pi_{Z_i}) f\| \leq \lambda_1 \|\omega_i \pi_{W_i} f\| + \lambda_2 \|z_i \pi_{Z_i}\| + \varepsilon \omega_i \|K^* f\|,$$

for all $i \in I$ and $f \in \mathcal{H}$. such that

$$\varepsilon < \frac{(1-\lambda_1)\sqrt{A}}{\|K\|(\sum_{i\in I}\omega_i^2)^{1/2}}$$

Then Z is a K-fusion frame for $\mathcal{H}$.

Finally, in the following theorem we show that under some small perturbations, K-duals of a K-fusion frame turn to the approximate K-dual for perturbed K-fusion frame.

Theorem 3.2. *Let $W = \{(W_i, \omega_i)\}_{i\in I}$ be a K-fusion frame for $\mathcal{H}$ with bounds A and B, respectively. Also, let $Z_i \subseteq \mathcal{H}$ be closed subspace of $\mathcal{H}$ for all i. If $\varepsilon > 0$ such that*

$$\|(T_W^* - T_Z^*)f\| < \varepsilon\|K^* f\|.$$

(i) If $0 < \varepsilon < \sqrt{A}$, then $Z = \{(Z_i, \omega_i)\}_{i\in I}$ is a K-fusion frame with bounds $(\sqrt{A} - \varepsilon)$ and $(\sqrt{B} + \varepsilon\|K\|)$, respectively.

(ii) If $\varepsilon > 0$ is sufficiently small, then every K-dual $V = \{(V_i, \upsilon_i)\}_{i\in I}$ of W satisfies $\|K - T_Z\phi_{vz}T_V^*\| < 1$.

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