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U-cross Gram matrixes and their associated reconstructions

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Abstract

In this paper, we study U-cross Gram matrixes, which can be produced by frames and Riesz bases, and their properties. Then we investigate some necessary or sufficient conditions for invertibility of this matrixes and try to reconstruct the elements of ℓ^2. It is important in application to state the inverse of U-cross Gram matrixes as the form of U-cross Gram matrixes.

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1. Introduction

In modern's life, many applications in industry refer to signal processing and it's ground is frame theory. In some areas of applications for example, sound [3] and physics [1] there are many problems that can be stated as operator theory. We know that an operator U can be stated by orthonormal basis $\{e_i\}_{i\in I}$ and described by a matrix such that all it's entries constructed by frames and their canonical dual, see [4]. Finally by a good extension an operator developed for Bessel sequences, frames and Riesz basis by Balazs, [2].

Throughout this paper $\mathcal{H}$ denotes a separable Hilbert space and I a countable indexing set. The identity operator on $\mathcal{H}$ is denoted by $I_{\mathcal{H}}$.

A *Riesz basis* for $\mathcal{H}$ is a family of the form $\{Ue_i\}_{i\in I}$, where $\{e_i\}_{i\in I}$ is an orthonormal basis for $\mathcal{H}$ and $U : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.

A family $\{f_i\}_{i\in I}$ in $\mathcal{H}$ is a *frame* if there exists constants $A, B > 0$ such that

$$A\,\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\,\|f\|^2, \qquad (f \in \mathcal{H}). \tag{1}$$

A and B are *frame bounds*. If $\{f_i\}_{i\in I}$ satisfies in the right hand of (1), then it is called a *Bessel sequence*. We say that a sequence $\{f_i\}_{i\in I}$ in $\mathcal{H}$ a *frame sequence* if it is a frame for $\overline{span}\{f_i\}_{i\in I}$. For a Bessel sequence $\{f_i\}_{i\in I}$ defines the *synthesis operator*

$$T : \ell^2 \to \mathcal{H}, \quad \left(\{c_i\}_{i\in I} \mapsto \sum_{i\in I} c_i f_i\right).$$

* speaker

Its adjoint operator $T^*: \mathcal{H} \to \ell^2$, so called *analysis operator* is given by

$$T^* f = \{\langle f, f_i \rangle\}_{i \in I}, \qquad (f \in \mathcal{H}).$$

For a frame $\{f_i\}_{i \in I}$, $S : \mathcal{H} \to \mathcal{H}$, which is defined by $Sf = TT^*f = \sum_{i \in I} \langle f, f_i \rangle f_i$, for all $f \in \mathcal{H}$, is called the *frame operator*.

Proposition 1.1. *For a sequence $\{f_i\}_{i \in I}$ in $\mathcal{H}$, the following conditions are equivalent:*

1. *$\{f_i\}_{i \in I}$ is a Riesz basis for $\mathcal{H}$.*
2. *$\{f_i\}_{i \in I}$ is complete in $\mathcal{H}$ and there exist constants $A, B > 0$ such that for every finite scalar sequence $\{c_i\}_{i \in I}$, one has*

$$A \sum_{i \in I} |c_i|^2 \leq \left\| \sum_{i \in I} c_i f_i \right\|^2 \leq \sum_{i \in I} |c_i|^2 . \tag{2}$$

2. Main Results

Definition 2.1. *Let $\Psi = \{\psi_i\}_{i \in I}$ be a Bessel sequence in $\mathcal{H}_1$ and $\Phi = \{\phi_i\}_{i \in I}$ a Bessel sequences in $\mathcal{H}_2$. For $U \in B(\mathcal{H}_1, \mathcal{H}_2)$, we call $\mathbf{G}_{U,\Phi,\Psi}$ given by*

$$(\mathbf{G}_{U,\Phi,\Psi})_{i,j} = \left\langle U\psi_j, \phi_i \right\rangle, \qquad (i, j \in I), \tag{3}$$

U-cross Gram matrix. If $\mathcal{H}_1 = \mathcal{H}_2$ and $U = I_{\mathcal{H}_1}$ it is called cross Gram matrix and denote it by $\mathbf{G}_{\Phi,\Psi}$. Also, if $\Phi = \Psi$ then this matrix is Gram matrix $\mathbf{G}_\Psi$.

Let $\Phi = \{\phi_i\}_{i \in I}$ and $\Psi = \{\psi_i\}_{i \in I}$ be two Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$ with upper bounds B and B', respectively. For all $U \in B(\mathcal{H}_1, \mathcal{H}_2)$, the following assertions hold.

1. The U-cross Gram matrix $\mathbf{G}_{U,\Phi,\Psi}$ defines a bounded operator from ℓ^2 to ℓ^2 and $\left\|\mathbf{G}_{U,\Phi,\Psi}\right\| \leq \sqrt{BB'}\|U\|$. Furthermore,

$$\mathbf{G}_{U,\Phi,\Psi} = T_\Phi^* U T_\Psi.$$

2. $(\mathbf{G}_{U,\Phi,\Psi})^* = \mathbf{G}_{U^*,\Psi,\Phi}$.

Proposition 2.2. *Let Φ, Ψ and Ξ be Bessel sequences in $\mathcal{H}$. Also $U_1, U_2 \in B(\mathcal{H})$. If $\Psi^\dagger$ is every dual of Ψ, then*

1. $\mathbf{G}_{U_1,\Phi,\Psi}\mathbf{G}_{U_2,\Psi^\dagger,\Xi} = \mathbf{G}_{U_1,\Phi,\Psi^\dagger}\mathbf{G}_{U_2,\Psi,\Xi} = \mathbf{G}_{U_1U_2,\Phi,\Xi}$.
2. $\mathbf{G}_{U_1,\Phi,\Psi}\mathbf{G}_{U_2,\Psi,\Xi} = \mathbf{G}_{U_1 S_\Psi U_2,\Phi,\Xi}$.

Theorem 2.3. *Let $\Psi = \{\psi\}_{i \in I}$ be a Riesz bases in $\mathcal{H}$ and $\Delta = \{\delta_i\}_{i \in I}$ the orthonormal basis of ℓ^2. Then $\mathbf{G}_{T_\Psi^*,\Delta,\widetilde{\Psi}}$, $\mathbf{G}_{S_\Psi,\widetilde{\Psi},\widetilde{\Psi}}$ and $\mathbf{G}_{S_\Psi^{-1},\Psi,\Psi}$ are identity matrixes.*

In the sequel, we find an inverse for U-cross Gram matrix under some conditions.

Theorem 2.4. *If Φ and Ψ be two Riesz basis and $U \in B(\mathcal{H})$ is an invertible operator then U-cross Gram matrix $\mathbf{G}_{U,\Phi,\Psi}$ has inverse and*

$$(\mathbf{G}_{U,\Phi,\Psi})^{-1} = \mathbf{G}_{U^{-1},\widetilde{\Psi},\widetilde{\Phi}}.$$

In the following, we state some sufficient conditions for the invertibility of a U-cross Gram matrix.

Theorem 2.5. *Let $\Psi = \{\psi_i\}_{i\in I}$ be a Bessel sequence and $\Phi = \{\phi\}_{i\in I}$ a Riesz basis with bounds A and B, respectively. Then $\mathbf{G}_{U,\Phi,\Psi}$ is invertible, if*

$$\sum_{i\in I} \|U\psi_i - \phi_i\| \leq \frac{B}{\sqrt{B_\Phi}},$$

where B_Φ is a Bessel bound of Φ.

Corollary 2.6. *Suppose that Φ and Ψ are two Bessel sequences in $\mathcal{H}$ with Bessel bounds B_Φ and B_Ψ, respectively. Also, $\mathbf{G}_{U,\Phi,\Psi}$ is invertible.*

1. *If $V \in B(\mathcal{H})$ such that*

$$\|U - V\| \leq \frac{1}{\left\|\mathbf{G}_{U,\Phi,\Psi}^{-1}\right\| \sqrt{B_\Phi B_\Psi}},$$

then $\mathbf{G}_{V,\Phi,\Psi}$ is invertible.

2. *If $\Xi = \{\xi_i\}_{i\in I}$ is a Bessel sequence in $\mathcal{H}$ such that*

$$\sum_{i\in I} \|\psi_i - \xi_i\| \leq \frac{1}{\left\|\mathbf{G}_{U,\Phi,\Psi}^{-1}\right\| \sqrt{B_\Phi}\, \|U\|},$$

then $\mathbf{G}_{U,\Phi,\Xi}$ is invertible.

3. *If $\Theta = \{\theta_i\}_{i\in I}$ is a Bessel sequence in $\mathcal{H}$ such that*

$$\sum_{i\in I} \|\phi_i - \theta_i\| \leq \frac{1}{\left\|\mathbf{G}_{U,\Phi,\Psi}^{-1}\right\| \sqrt{B_\Psi}\, \|U\|},$$

then $\mathbf{G}_{U,\Theta,\Psi}$ is invertible.

Theorem 2.7. *Let $\mathbf{G}_{U,\Phi,\Psi}$ be invertible. Then Ψ and Φ are Riesz sequences.*

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