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Drazin and Moore-Penrose spectrum

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Abstract

In this paper, we are going to define and study the Drazin and Moore-Penrose spectrum and obtain some result for Drazin and Moore-Penrose inverse.

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1. Introduction

An element of a complex Banach algebra A is called regular (or relatively regular) if there is $x \in A$ such that $axa = a$. If a is relatively regular, then it has generalized inverse, which is an element $b \in A$ satisfying the equations $aba = a$ and $bab = b$ .A relation between a relatively regular element and its generalized inverse is reflexive in the sense that if b is a generalized inverse of a , then a is a generalized inverse of b.

We say that an element $a \in A$ is Drazin invertible if there is $x \in A$ such that

$$xa = ax, \qquad ax^2 = x, \qquad a^k = a^{k+1}x \tag{1}$$

for some nonnegative integer K. For any Drazin invertible $a \in A$ such that x is unique; We write $x = a^D$, and call it the Drazin inverse of a. We write A^D for the set of all Drazin invertible elements of A.

An element $a \in A$ is Moore-Penrose invertible if there exists $x \in A$ such that

$$xax = x, \qquad axa = a, \qquad (ax)^* = ax, \qquad (xa)^* = xa \tag{2}$$

There is at most one element x satisfying above; If a is Moore-Penrose invertible, the unique solution of (1.2) is called the Moore-Penrose inverse of a and is denoted by $a^\dagger$. The set of all Moore-Penrose invertible elements of A is denoted by $A^\dagger$.

Let A be a unital C^*- algebra. The Drazin spectrum of an element $a \in A$ is the set

$$sp_{DR}(a) = \{\lambda \in C; \lambda e - a \text{ is not Drazin invertible}\}.$$

Let A be a unital C^*- algebra. The Moore-Penrose spectrum of an element $a \in A$ is the set

$$sp_{MP}(a) = \{\lambda \in C; \lambda e - a \text{ is not Moore-Penrose invertible}\}.$$

* speaker

2. Main results

Now we state some properties of Drazin and Moore-Penrose spectrum. To achieve our goal, we need to express the following theorems.

Theorem 2.1. *Consider C^*-algebra A, and $a, x \in A$. Then*
(i) the equation $a = axa$ and $(ax)^* = ax$ are equivalent to $a = x^*a^*a$.
(ii) the equation $a = axa$ and $(xa)^* = xa$ are equivalent to $a = aa^*x^*$.
(iii) the equation $x = xax$ and $(ax)^* = ax$ are equivalent to $x = xx^*a^*$.
(iv)the equation $x = xax$ and $(xa)^* = xa$ are equivalent to $x = a^*x^*x$.

Theorem 2.2. *Consider C^*-algebra A, and $a \in A$. Then the following statements are equivalent:*
(i) $x \in A$ is the Moore-Penrose inverse of a.
(ii)$a^* = xaa^*$ and $x = xx^*a^*$.
(iii) $a = aa^*x^*$ and $x^* = axx^*$.
(iv) $a^* = a^*ax$ and $x = a^*x^*x$.
(v) $a = x^*a^*a$ and $x^* = x^*xa$.

Theorem 2.3. *Let $a \in A$ be normal. Then the following are true.*
(i) $a \in A^\dagger \iff a \in A^D$.
(ii) if $a \in A^\dagger$, then $a^\dagger$ is normal and commute with a.

Theorem 2.4. *An element a of a C^* -algebra A is Moore-Penrose invertible if and only if a^*a (respectively aa^*) is Drazin invertible. If $a \in A^\dagger$, then*
$$a^\dagger = (a^*a)^D a^* = a^*(aa^*)^D$$

Theorem 2.5. *An element of a C^* -algebra is Moore-Penrose invertible if and only if it is regular*

Theorem 2.6. *An element of a C^* -algebra is Moore-Penrose invertible if and only if it is regular*

Theorem 2.7. *Let $a, b \in A^\dagger$, Then*
$$b^\dagger - a^\dagger = -b^\dagger(b-a)a^\dagger + (e - b^\dagger b)(b^* - a^*)(a^\dagger)^*a^\dagger + b^\dagger(b^\dagger)^*(b^* - a^*)(e - aa^\dagger) \quad (3)$$

Theorem 2.8. *If $a, b \in A^\dagger$ are such that $\|b - a\| < \frac{1}{2}\|a^\dagger\|^{-1}$ and $\|bb^\dagger - aa^\dagger\| < 1$ then*
$$\|b^\dagger\| \leq 4\|a^\dagger\| \quad (4)$$

Theorem 2.9. *Let a_n, a be nonzero elements of $A^\dagger$ such that $a_n \longrightarrow a$ in A . Then the following conditions are equivalent*
$$a_n^\dagger \longrightarrow a^\dagger \quad (5)$$
$$a_n a_n^\dagger \longrightarrow aa^\dagger \quad (6)$$

$$a_n^{\dagger}a_n \longrightarrow a^{\dagger}a \tag{7}$$

$$sup\|a_n^{\dagger}\| < \infty. \tag{8}$$

Theorem 2.10. *Let $a(t)$ be a C^* -algebra valued function defined on an interval J such that $0 \neq a(t) \in A^{\dagger}$ for all $t \in J$ and that $a(t)$ is defferentiable at t_0 . Then the function $a^{\dagger}(t)$ is differentiable at t_0 if and only if one of the condition of perivious theorem is satisfied. the derivative $(a^{\dagger})' = (a^{\dagger})'(t_0)$ is given by*

$$(a^{\dagger})' = -a^{\dagger}a'a^{\dagger} + (e - a^{\dagger}a)(a')^*(a^{\dagger})^*a^{\dagger} + a^{\dagger}(a^{\dagger})^*(a')^*(e - aa^{\dagger}) \tag{9}$$

where $a, a^*, a^{\dagger}, a'$ stand for $a(t_0), a^*(t_0), a^{\dagger}(t_0), a'(t_0)$ respectively.

Theorem 2.11. *Let $a(t)$ be a C^* -algebra valued function defined on an interval J such that $0 \neq a(t) \in A^{\dagger}$ for all $t \in J$ and that $a(t)$ is defferentiable at t_0 . Then the function $a^{\dagger}(t)$ is continuous at t_0. The following conditions are equivalent.*
$a^{\dagger}(t)$is continuous at t_0.
$a(t)a^{\dagger}(t)$is continuous at t_0.
$a^{\dagger}(t)a(t)$is continuous at t_0.
there is $\delta > 0$ such that sup $\|a^{\dagger}(t)\| < \infty$.

In the following theorems we state two properties of Drazin spectrum.

Theorem 2.12. *Let A be unital Banach algebra and consider $a \in A$. Then the following statements are equivalent.*
(i) $\sigma(a)$ is at most countable.
(ii) $\sigma_{DR}(a)$ is at most countable.

Theorem 2.13. *Let A be a unital Banach algebra and consider $a, b \in A$. Then*

$$\sigma_{DR}(ab) = \sigma_{DR}(ba).$$

Now by above theorems we stablish a new properties of Moore-Penrose spectrum

Theorem 2.14. *sp_{MP} is non-empty.*

Is it true that if we have a Banach algebra A that every element of A has Moore-Penrose inverse then $A \simeq C$?

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