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Drazin and Moore-Penrose spectrum

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Abstract

In this paper, we are going to define and study the Drazin and Moore-Penrose spectrum and obtain some result for Drazin and Moore-Penrose inverse.

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1. Introduction

An element of a complex Banach algebra A is called regular (or relatively regular) if there is $x \in A$ such that axa = a. If a is relatively regular, then it has generalized inverse, which is an element $b \in A$ satisfying the equations aba = a and bab = b. A relation between a relatively regular element and its generalized inverse is reflexive in the sense that if b is a generalized inverse of a, then a is a generalized inverse of b.

We say that an element $a \in A$ is Drazin invertible if there is $x \in A$ such that

$$xa = ax, \qquad ax^2 = x, \qquad a^k = a^{k+1}x \tag{1}$$

for some nonnegetive integer K. For any Drazin invertible $a \in A$ such that x is unique; We write $x = a^D$, and call it the Drazin inverse of a. We write A^D for the set of all Drazin invertible elements of A.

An element $a \in A$ is Moore-Penrose invertible if there exists $x \in A$ such that

$$xax = x,$$
 $axa = a,$ $(ax)^* = ax,$ $(xa)^* = xa$ (2)

There is at most one element x satisfying above; If a is Moore-Penrose invertible, the unique solution of (1.2) is called the Moore-Penrose inverse of a and is denoted by a^{\dagger} . The set of all Moore-Penrose invertible elements of A is denoted by A^{\dagger} .

Let A be a unital C^* - algebra. The Drazin spectrum of an element $a \in A$ is the set

 $sp_{DR}(a) = \{\lambda \in C; \lambda e - a \text{ is not Drazin invertible}\}.$

Let A be a unital C^* - algebra. The Moore-Penrose spectrum of an element $a \in A$ is the set

 $sp_{MP}(a) = \{\lambda \in C; \lambda e - a \text{ is not Moore-Penrose invertible}\}.$

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2. Main results

Now we state some properties of Drazin and Moore-Penrose spectrum. To achieve our goal, we need to express the following theorems.

Theorem 2.1. Consider C^* -algebra A, and $a, x \in A$. Then (i) the equation a = axa and $(ax)^* = ax$ are equivalent to $a = x^*a^*a$. (ii) the equation a = axa and $(xa)^* = xa$ are equivalent to $a = aa^*x^*$. (iii) the equation x = xax and $(ax)^* = ax$ are equivalent to $x = xx^*a^*$. (iv) the equation x = xax and $(xa)^* = xa$ are equivalent to $x = a^*x^*x$.

Theorem 2.2. Consider C^* -algebra A, and $a \in A$. Then the following statements are equivalent:

(i) $x \in A$ is the Moore-Penrose inverse of a. (ii) $a^* = xaa^*$ and $x = xx^*a^*$. (iii) $a = aa^*x^*$ and $x^* = axx^*$. (iv) $a^* = a^*ax$ and $x = a^*x^*x$. (v) $a = x^*a^*a$ and $x^* = x^*xa$.

Theorem 2.3. Let $a \in A$ be normal. Then the following are true. (i) $a \in A^{\dagger} \iff a \in A^{D}$. (ii) if $a \in A^{\dagger}$, then a^{\dagger} is normal and commute with a.

Theorem 2.4. An element *a* of *a C*^{*} -algebra *A* is Moore-Penrose invertible if and only if a^*a (respectively aa^*) is Drazin invertible. If $a \in A^{\dagger}$, then $a^{\dagger} = (a^*a)^D a^* = a^*(aa^*)^D$

Theorem 2.5. An element of a C^* -algebra is Moore-Penrose invertible if and only if it is regular

Theorem 2.6. An element of a C^* -algebra is Moore-Penrose invertible if and only if it is regular

Theorem 2.7. Let $a, b \in A^{\dagger}$, Then

$$b^{\dagger} - a^{\dagger} = -b^{\dagger}(b - a)a^{\dagger} + (e - b^{\dagger}b)(b^{*} - a^{*})(a^{\dagger})^{*}a^{\dagger} + b^{\dagger}(b^{\dagger})^{*}(b^{*} - a^{*})(e - aa^{\dagger})$$
(3)

Theorem 2.8. If $a, b \in A^{\dagger}$ are such that $||b - a|| < \frac{1}{2} ||a^{\dagger}||^{-1}$ and $||bb^{\dagger} - aa^{\dagger}|| < 1$ then

$$\|b^{\dagger}\| \le 4\|a^{\dagger}\| \tag{4}$$

Theorem 2.9. Let a_n , a be nonzero elements of A^{\dagger} such that $a_n \longrightarrow a$ in A. Then the following conditions are equivalent

$$a_n^{\dagger} \longrightarrow a^{\dagger}$$
 (5)

$$a_n a_n^\dagger \longrightarrow a a^\dagger$$
 (6)

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$$a_n^{\dagger} a_n \longrightarrow a^{\dagger} a$$
 (7)

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$$\sup \|a_n^{\dagger}\| < \infty. \tag{8}$$

Theorem 2.10. Let a(t) be a C^* -algebra valued function defined on an interval J such that $0 \neq a(t) \in A^{\dagger}$ for all $t \in J$ and that a(t) is defineratiable at t_0 . Then the function $a^{\dagger}(t)$ is differentiable at t_0 if and only if one of the condition of perivious theorem is satisfied. the derivative $(a^{\dagger})' = (a^{\dagger})'(t_0)$ is given by

$$(a^{\dagger})' = -a^{\dagger}a'a^{\dagger} + (e - a^{\dagger}a)(a')^{*}(a^{\dagger})^{*}a^{\dagger} + a^{\dagger}(a^{\dagger})^{*}(a')^{*}(e - aa^{\dagger})$$
(9)

where a, a^*, a^{\dagger}, a' stand for $a(t_0), a^*(t_0), a^{\dagger}(t_0), a'(t_0)$ respectively.

Theorem 2.11. Let a(t) be a C^* -algebra valued function defined on an interval J such that $0 \neq a(t) \in A^{\dagger}$ for all $t \in J$ and that a(t) is defineration at t_0 . Then the function $a^{\dagger}(t)$ is continuous at t_0 . The following conditions are equivalent.

 $a^{\dagger}(t)$ is continuous at t_0 . $a(t)a^{\dagger}(t)$ is continuous at t_0 . $a^{\dagger}(t)a(t)$ is continuous at t_0 . there is $\delta > 0$ such that sup $||a^{\dagger}(t)|| < \infty$.

In the following theorems we state two properties of Drazin spectrum.

Theorem 2.12. Let A be unital Banach algebra and consider $a \in A$. Then the following statements are equivalent.

(i) $\sigma(a)$ is at most countable.

(ii) $\sigma_{DR}(a)$ is at most countable.

Theorem 2.13. Let A be a unital Banach algebra and consider $a, b \in A$. Then

$$\sigma_{DR}(ab) = \sigma_{DR}(ba)$$

Now by above theorems we stablish a new properties of Moore-Penrose spectrum

Theorem 2.14. sp_{MP} is non-empty.

Is it true that if we have a Banach algebra A *that every element of* A *has Moore-Penrose inverse then* $A \simeq C$?

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