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Transitivity Property and Extended Weil Formula

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Abstract

For a locally compact group G and two closed subgroups H, N of G, we are going to write $\mu_{G/H}$ as the integral, with respect to $\mu_{G/N}$ of a family of measures on G/H indexed by the points of G/N, in which $\mu_{G/H}$ and $\mu_{G/N}$ are measures on quotient spaces G/H and G/N, respectively.

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1. Introduction

Let G be a locally compact group and H be a closed subgroup of G with left Haar measures λ and λ_H, respectively. Consider G/H as a quotient space on which G acts from the left and a Radon measure μ on G/H is said to be G-invariant if $\mu_x(yH) = \mu(yH)$ for all $x, y \in G$, where μ_x is defined by $\mu_x(E) = \mu(xE)$ (for Borel subsets E of G/H). It is well known that there is a G-invariant Radon measure μ on G/H if and only if $\Delta_G|_H = \Delta_H$, where Δ_G, Δ_H are the modular functions on G and H, respectively. In this case we have,

$$\int_G f(x)d\lambda(x) = \int_{G/H} Pf(xH)d\mu(xH) = \int_{G/H}\int_H f(xh)d\lambda_H(h)d\mu(xH), \qquad (1)$$

in which $Pf(xH) = \int_H f(xh)d\lambda_H(h)$ is a continuous linear map from $C_c(G)$ onto $C_c(G/H)$.
The formula (1) is called Weil formula. It has been shown that if λ be a positive Radon measure on G such that

$$d\lambda(xh) = \Delta_H(h)d\lambda(x),$$

for $h \in H$, one can then form a unique positive Radon measure $\mu_{G/H}$ on quotient space G/H (For more details see [3, 5, 6]).

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2. Main result

Let G be a locally compact group and H, N be two closed subgroups of G such that $H \subseteq N$ and $\Delta_N|_H = \Delta_H$. We denote by q_N, q_H, p, the canonical mappings of G onto G/N, of G onto G/H and of N onto N/H. Let λ_N and λ_H be the left Haar measures on N and H, respectively. Since $\Delta_N|_H = \Delta_H$, then there exists a G-invariant measure $\mu_{N/H}$ on N/H . On the otherhand, if λ is a left Haar measure on G such that $d\lambda(xh) = \Delta_H(h)d\lambda(x)$, one can find Radon measures $\mu_{G/N}$, $\mu_{G/H}$ on quotient spaces G/N and G/H. Then it is easy to check that, the mapping $(x, n) \mapsto q_H(xn)$ of $G \times N$ into G/H is continuous. Since $q_H(xnh) = q_H(xn)$, for all $h \in H$, this mapping defines a continuous mapping of $G \times (N/H)$ into G/H. Whence for each fixed $x \in G$, the mapping ψ_x of N into G such that $\psi_x(n) = xn$, defines a mapping ω_x of N/H into G/H in which

$$\omega_x(p(n)) = q_H(\psi_x(n)) = q_H(xn).$$

It is easy to show that $\psi_{xn} = \psi_x o \varrho_N(n)$, thererfore that $\omega_{xn} = \omega_x o \varrho_{N/H}(n)$, for all $n \in N$, in which $\varrho_N(n)(n') = nn'$. The following lemma shows that the map ω_x is proper.

Lemma 2.1. *Let E be a compact subset of G/H and K be a compact subset of G. Then $\cup_{x\in K}\omega_x^{-1}(E)$ is relatively compact in N/H. In particular, $\cup_{x\in K}\omega_x^{-1}(E)$ is contained in a compact subset of N/H.*

PROOF. Let F be a compact subset of G such that $q_H(F) = E$. Let L be the set of $n \in N$ such that Kn intersects F. Then L is compact (see [1], ChapterIII,$\oint$4.5, Theorem1). Let $n \in N$, such that $p(n) \in \cup_{x\in K}\omega_x^{-1}(E)$. Thus there exists $x \in K$ such that $\omega_x(p(n)) \in E$. i.e. $q_H(xn) \in E$ and since $q_H(F) = E$, there exists $h \in H$, $xnh \in F$. Then $nh \in L$. So $p(nh) = p(n) \in p(L)$. That is $\cup_{x\in K}\omega_x^{-1}(E) \subseteq p(L)$. □

Let $\mathcal{M}(N/H)$ and $\mathcal{M}(G/H)$ be complex measure spaces on quotient spaces N/H and G/H, respectively, as introduced in [6]. Lemma 2.1 shows that the mapping ω_x is proper. Then ω_x extends continuously to a map from $\mathcal{M}(N/H)$ into $\mathcal{M}(G/H)$ ([4], Section 4.5).

Now let $\varphi \in C_c(G/H)$. Define the function Ψ of G into $\mathcal{M}(G/H)$ such that

$$\Psi(x) = \langle \varphi, \omega_x(\mu_{N/H}) \rangle = \int_{N/H} \varphi(\omega_x(p(n))d\mu_{N/H}(p(n)).$$

The function Ψ is continuous and compact support. Moreover, since the measure $\mu_{N/H}$ is G-invariant, we have

$$\begin{aligned}
\Psi(xn) &= \langle\varphi, \omega_{xn}(\mu_{N/H})\rangle \\
&= \langle\varphi, \omega_x o \varrho_{N/H}(n)(\mu_{N/H})\rangle \\
&= \int_{N/H} \varphi(\omega_x o \varrho_{N/H}(n)(p(n')))d\mu_{N/H}(p(n')) \\
&= \int_{N/H} \varphi(\omega_x(nn'H))d\mu_{N/H}(p(n')) \\
&= \int_{N/H} \varphi(\omega_x(n'H))d\mu_{N/H}(p(n')) \\
&= \langle\varphi, \omega_x(\mu_{N/H})\rangle \\
&= \Psi(x),
\end{aligned}$$

for $n \in N$. Then the mapping $\tilde{\Psi}$ of G/N into $\mathcal{M}(G/H)$ in which

$$\tilde{\Psi}(q_N(x)) = \langle\varphi, \omega_x(\mu_{N/H})\rangle,$$

is continuous with compact support, for all $\varphi \in C_c(G/H)$.

Proposition 2.2. *Let $\varphi \in C_c(G/H)$. Then*

$$\int_{G/N} \langle\varphi, \omega_x(\mu_{N/H})\rangle d\mu_{G/N}(q_N(x)) = \int_{G/H} \varphi(q_H(x))d\mu_{G/H}(q_H(x)).$$

Proof. By (1), for $\varphi \in C_c(G/H)$ we have

$$\int_{G/H} \varphi(q_H(x))d\mu_{G/H}(q_H(x)) = \int_G f(x)d\lambda(x),$$

where $\varphi = Pf$ and $f \in C_c(G)$. Also,

$$\begin{aligned}
&\int_{G/N} \langle\varphi, \omega_x(\mu_{N/H})\rangle d\mu_{G/N}(q_N(x)) = \\
&\int_{G/N} \int_{N/H} (Pf(\omega_x(p(n))d\mu_{N/H}(p(n))d\mu_{G/N}(q_N(x)) = \\
&\int_{G/N} \int_{N/H} (Pf(q_H(xn))d\mu_{N/H}(p(n))d\mu_{G/N}(q_N(x)) = \\
&\int_{G/N} \int_{N/H} \int_H L_x f(nh)d\lambda_H(h)d\mu_{N/H}(p(n))d\mu_{G/N}(q_N(x)) = \\
&\int_{G/N} \int_N f(xn)d\lambda_N(n)d\mu_{G/N}(q_N(x)) = \\
&\int_G f(x)d\lambda(x),
\end{aligned}$$

in which $L_x f(n) = f(xn)$.

□

Corollary 2.3. *(i) Let φ be a $\mu_{G/H}$-integrable function on G/H. There exists a $\mu_{G/N}$-negligible subset E of G/N having the following property: if $x \in G$ is such that $q_N(x) \notin E$, then the function $\varphi o \omega_x$ on N/H is $\mu_{N/H}$-integrable. The integral $\int_{N/H} \varphi(\omega_x(p(n))d\mu_{N/H}(p(n)$ is a $\mu_{G/N}$-integrable function and*

$$\int_{G/H} \varphi(q_H(x))d\mu_{G/H}(q_H(x)) = \int_{G/N} d\mu_{G/N}(q_N(x)) \int_{N/H} \varphi(\omega_x(p(n))d\mu_{N/H}(p(n)). \quad (2)$$

(ii) suppose that there exists a bounded positive measure $\mu_{G/H}$ on quotient space G/H. Then there exists a bounded positive measure on quotient space N/H.

PROOF. (i) By proposition 2.2 we have,

$$\int_{G/H} \varphi(q_H(x))d\mu_{G/H}(q_H(x)) =$$
$$\int_{G/N} \langle \varphi, \omega_x(\mu_{N/H}) \rangle d\mu_{G/N}(q_N(x)) =$$
$$\int_{G/N} \int_{N/H} (\varphi(\omega_x(p(n))d\mu_{N/H}(p(n))d\mu_{G/N}(q_N(x)).$$

(ii) The function 1 on G/H is $\mu_{G/H}$- integrable. By the part (i), the function 1 on N/H is $\mu_{N/H}$-integrable. Thus $\mu_{N/H}$ is bounded. □

Remark 2.4. *If $H = \{e\}$, then the Weil formula can be concluded from (2).*

References

[1] N. Bourbaki, *Elements of Mathematics, General Topology*, springer, Verlag Berlin, New york, 1971.

[2] F. Esmaeelzadeh, R. A. Kamyabi Gol *Homogeneous Spaces and Square Integrable Representations*,Ann. Funct. Anal. **7** (2016) , no 1, 9-16.

[3] G.B. Folland, *Real Analysis* , Modern techniques and their applications. 2nd ed. Wiley, New York, 1999.

[4] G.B. Folland, *A Course in Abstract Harmonic Analysis*, CRC Press, Boca Raton, FL, 1995.

[5] H. Reiter, J. Stegeman, *Classical Harmonic Analysis and Locally compact Group*, Claredon press, 2000.

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