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Comparison of the topological centers of a bilinear mapping and its third adjoint

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Abstract

Let $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ be a bilinear mapping on normed spaces. In this paper we investigate that are the topological centers of f, w^*-dense in the corresponding topological centers of its extensions f^{***} and f^{t***t}? we show that although it has positive answer on some special cases but this is not true in general.

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1. Introduction

According to [1] and [2] for every bounded bilinear mapping $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ (on normed spaces $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$) we have two natural extensions from $\mathcal{X}^{**} \times \mathcal{Y}^{**}$ to $\mathcal{Z}^{**}$. Also the definition of regularity of bilinear mappings mentioned in [1] and [2]. First of all We recall these definitions.

For a bounded bilinear mapping $f : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ we define the adjoint $f^* : \mathcal{Z}^* \times \mathcal{X} \to \mathcal{Y}^*$ of f by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle \qquad (x \in \mathcal{X}, y \in \mathcal{Y} \text{ and } z^* \in \mathcal{Z}^*).$$

Also this process may be repeated to define $f^{**} = (f^*)^* : \mathcal{Y}^{**} \times \mathcal{Z}^* \to \mathcal{X}^*$ and $f^{***} = (f^{**})^* : \mathcal{X}^{**} \times \mathcal{Y}^{**} \to \mathcal{Z}^{**}$. It can readily verified that f^{***} is the unique extension of f for which the maps

$$\cdot \mapsto f^{***}(\cdot, y^{**}), \quad \cdot \mapsto f^{***}(x, \cdot) \quad (x \in \mathcal{X}, y^{**} \in \mathcal{Y}^{**}),$$

are $w^* - w^*-$separately continuous.

Let f^t be the transpose of f, that is the bounded bilinear mapping $f^t : \mathcal{Y} \times \mathcal{X} \longrightarrow \mathcal{Z}$ defined by $f^t(y, x) = f(x, y)$ $(x \in \mathcal{X}, y \in \mathcal{Y})$. If we continue the latter process with f^t instead of f, we come to the bounded bilinear mapping $f^{t***t} : \mathcal{X}^{**} \times \mathcal{Y}^{**} \to \mathcal{Z}^{**}$, that is the unique extension of f for which the maps

$$\cdot \mapsto f^{t***t}(x^{**}, \cdot), \quad \cdot \mapsto f^{t***t}(\cdot, y) \quad (y \in \mathcal{Y}, x^{**} \in \mathcal{X}^{**}),$$

* speaker

are $w^* - w^*-$ continuous.
We define the left topological center $Z_\ell(f)$ by

$$\begin{aligned} Z_\ell(f) &= \{x^{**} \in \mathcal{X}^{**}; y^{**} \longrightarrow f^{***}(x^{**}, y^{**}) : \mathcal{Y}^{**} \longrightarrow \mathcal{Z}^{**} \text{ is } w^* - \text{continuous}\} \\ &= \{x^{**} \in \mathcal{X}^{**}; f^{***}(x^{**}, y^{**}) = f^{t***t}(x^{**}, y^{**}) \ \text{ for every } y^{**} \in \mathcal{Y}^{**}\}, \end{aligned}$$

and the right topological center $Z_r(f)$ of f by

$$\begin{aligned} Z_r(f) &= \{y^{**} \in \mathcal{Y}^{**}; x^{**} \longrightarrow f^{t***t}(x^{**}, y^{**}) : \mathcal{X}^{**} \longrightarrow \mathcal{Z}^{**} \text{ is } w^* - \text{continuous}\} \\ &= \{y^{**} \in \mathcal{Y}^{**}; f^{***}(x^{**}, y^{**}) = f^{t***t}(x^{**}, y^{**}) \ \text{ for every } x^{**} \in \mathcal{X}^{**}\}. \end{aligned}$$

Clearly, $\mathcal{X} \subseteq Z_\ell(f), \mathcal{Y} \subseteq Z_r(f)$ and $Z_r(f) = Z_\ell(f^t)$.

A bounded bilinear mapping f is said to be Arens regular if $f^{***} = f^{t***t}$. This is equivalent to $Z_\ell(f) = \mathcal{X}^{**}$ as well as $Z_r(f) = \mathcal{Y}^{**}$. The mapping f is said to be left (resp. right) strongly Arens irregular if $Z_\ell(f) = \mathcal{X}$ (resp. $Z_r(f) = \mathcal{Y}$).

We know that $\mathcal{X} \subseteq Z_\ell(f) \subseteq \mathcal{X}^{**} \subseteq Z_\ell(f^{***}) \subseteq \mathcal{X}^{****}$ and $\mathcal{Y} \subseteq Z_r(f) \subseteq \mathcal{Y}^{**} \subseteq Z_r(f^{***}) \subseteq \mathcal{Y}^{****}$ in general. In this paper we investigate the relationship of $\overline{Z_\ell(f)}^{w^*}$ with $Z_\ell(f^{***})$ and $Z_\ell(f^{t***t})$ and similarly for the right topological centers.

2. Main results

Theorem 2.1. *(i)* $\overline{Z_\ell(f)}^{w^*} \subseteq Z_\ell(f^{***})$ *if and only if* $\overline{Z_\ell(f)}^{w^*} \subseteq Z_\ell(f^{t***t})$

(ii) $\overline{Z_r(f)}^{w^*} \subseteq Z_r(f^{***})$ *if and only if* $\overline{Z_r(f)}^{w^*} \subseteq Z_r(f^{t***t})$

Corollary 2.2. *If f is Arens regular then f^{***} is Arens regular if and only if f^{t***t} is Arens regular.*

Corollary 2.3. *If f^{***} is Arens regular then* $\overline{Z_\ell(f)}^{w^*} \subseteq Z_\ell(f^{t***t})$, *and if f^{t***t} is Arens regular then* $\overline{Z_\ell(f)}^{w^*} \subseteq Z_\ell(f^{***})$.

Theorem 2.1 says that it is sufficient to investigate only the relationship of the topological centers f^{***} and w^*-cluster of the topological centers of f.

Also it is easy to see that if $\mathcal{X}$ is reflexive then $\overline{Z_\ell(f)}^{w^*} = \mathcal{X} = Z_\ell(f^{***})$ and if $\mathcal{Y}$ is reflexive then $\overline{Z_r(f)}^{w^*} = \mathcal{X} = Z_r(f^{***})$. So we assume that $\mathcal{X}$ and $\mathcal{Y}$ are not reflexive. On the other hand in [3] it is shown that there is an Arens regular bilinear mapping f such that f^{***} is not Arens regular. Therefore in this case $Z_\ell(f^{***}) \subsetneq \overline{Z_\ell(f)}^{w^*}$ and the equality are not valid in general.
In the sequel we investigate the relationship $\overline{Z_\ell(f)}^{w^*}$ with $Z_\ell(f^{***})$ and $\overline{Z_r(f)}^{w^*}$ with $Z_r(f^{***})$ in special cases. The following theorem has a proof similar to the proof of theorem 2.1.

Theorem 2.4. *(i) For each* $x^{****} \in \overline{Z_\ell(f)}^{w^*}$ *and* $y^{****} \in \mathcal{Y}^{****}$,

$$f^{******}(x^{****}, y^{****}) = f^{t***t***}(x^{****}, y^{****})$$

and

$$f^{t*****t}(x^{****}, y^{****}) = f^{***t***t}(x^{****}, y^{****}).$$

(ii) For each $x^{****} \in Z_\ell(f^{***})$ *and* $y^{****} \in \mathcal{Y}^{****}$,

$$f^{******}(x^{****}, y^{****}) = f^{***t***t}(x^{****}, y^{****}).$$

(iii) For each $y^{****} \in \overline{Z_r(f)}^{w^*}$ *and* $x^{****} \in \mathcal{X}^{****}$,

$$f^{******}(x^{****}, y^{****}) = f^{t***t***}(x^{****}, y^{****})$$

and

$$f^{t*****t}(x^{****}, y^{****}) = f^{***t***t}(x^{****}, y^{****})$$

(iv) For each $y^{****} \in Z_r(f^{***})$ *and* $x^{****} \in \mathcal{X}^{****}$,

$$f^{******}(x^{****}, y^{****}) = f^{***t***t}(x^{****}, y^{****}).$$

Corollary 2.5. *(i)* $f^{******}|_{\overline{Z_\ell(f)}^{w^*}\times\mathcal{Y}^{****}} = f^{t*****t}|_{\overline{Z_\ell(f)}^{w^*}\times\mathcal{Y}^{****}}$ *if and only if* $\overline{Z_\ell(f)}^{w^*} \subseteq Z_\ell(f^{***})$ *if and only if* $f^{***t***t}|_{\overline{Z_\ell(f)}^{w^*}\times\mathcal{Y}^{****}} = f^{t***t***}|_{\overline{Z_\ell(f)}^{w^*}\times\mathcal{Y}^{****}}$.

(ii) $f^{******}|_{\mathcal{X}^{****}\times\overline{Z_r(f)}^{w^*}} = f^{t*****t}|_{\mathcal{X}^{****}\times\overline{Z_r(f)}^{w^*}}$ *if and only if* $\overline{Z_r(f)}^{w^*} \subseteq Z_r(f^{***})$ *if and only if* $f^{***t***t}|_{\mathcal{X}^{****}\times\overline{Z_r(f)}^{w^*}} = f^{t***t***}|_{\mathcal{X}^{****}\times\overline{Z_r(f)}^{w^*}}$.

(iii) If $Z_\ell(f^{***}) \subseteq \overline{Z_\ell(f)}^{w^*}$ *then* $f^{***t***t}|_{Z_\ell(f^{***})\times\mathcal{Y}^{****}} = f^{t***t***}|_{Z_\ell(f^{***})\times\mathcal{Y}^{****}}$ *and*

$$f^{******}|_{Z_\ell(f^{***})\times\mathcal{Y}^{****}} = f^{t*****t}|_{Z_\ell(f^{***})\times\mathcal{Y}^{****}}.$$

(iv) If $Z_r(f^{***}) \subseteq \overline{Z_r(f)}^{w^*}$ *then* $f^{***t***t}|_{\mathcal{X}^{****}\times Z_r(f^{***})} = f^{t***t***}|_{\mathcal{X}^{****}\times Z_r(f^{***})}$ *and*

$$f^{******}|_{\mathcal{X}^{****}\times Z_r(f^{***})} = f^{t*****t}|_{\mathcal{X}^{****}\times Z_r(f^{***})}.$$

Corollary 2.6. *If* $f^{t***t***} = f^{***t***t}$, *then* $\overline{Z_\ell(f)}^{w^*} \subseteq Z_\ell(f^{t***t})$ *and* $\overline{Z_r(f)}^{w^*} \subseteq Z_r(f^{t***t})$

on the other hand by two routine w^*−limit, we have the following proposition.

Proposition 2.7. *If* $\mathcal{X}^{**} \subseteq \overline{Z_\ell(f)}^{w^*}$ *then* $Z_\ell(f^{***}) \subseteq \overline{Z_\ell(f)}^{w^*}$.

Note that if f is Arens regular, then $Z_\ell(f^{***}) \subseteq \overline{Z_\ell(f)}^{w^*}$ and if it is strongly Arens irregular then it maybe $Z_\ell(f^{***}) \nsubseteq \overline{Z_\ell(f)}^{w^*}$.

References

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