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Cardinality of the set of topologically left invariant means on left uniformly continuous functionals

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Abstract

For a Lau algebra $\mathcal{A}$ with a certain condition we show that the number of topologically left invariant means on $\mathcal{A}^*$ is equal to the number of topologically left invariant means on LUC$(\mathcal{A}^*)$. Using this for a nondiscrete locally comapct group G we prove that the cardinality of the set of topologically invariant means on $UC(\widehat{G})$ is equal to $2^{2^{b(G)}}$. In particular, G is discrete if and only if there is a unique topologically invariant mean on $UC(\widehat{G})$.

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1. Introduction

In [2] Lau introduced and investigated a nice family of Banach algebras under the name F-algebras; later, F-algebras were termed Lau algebras. In fact, a Banach algebra $\mathcal{A}$ is called a *Lau algebra* if the dual space $\mathcal{A}^*$ of $\mathcal{A}$ is a von Neumann algebra and the identity element, I of $\mathcal{A}^*$ is a multiplicative linear functional on $\mathcal{A}$.

Examples of Lau algebras are the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group G; see [2]. It also includes the Fourier-Stieltjes algebra $B(G)$ of a topological group G. Moreover, the hypergroup algebra $L^1(H)$ and the measure algebra $M(H)$ of a locally compact hypergroup H with a left Haar measure are Lau algebras. A particular example of Lau algebras is the quantum group algebra $L^1(\mathbb{G})$ of a locally compact quantum group $\mathbb{G}$.

The main purpose of this paper is to show that for a Lau algebra $\mathcal{A}$ satisfying $\langle \mathcal{A}\mathcal{A}_0\rangle = \mathcal{A}_0$, the cardinality of the set of topologically left invariant means on $\mathcal{A}^*$ is equal to the cardinality of the set of topologically left invariant means on LUC$(\mathcal{A}^*)$. As an application of this result we show that for a nondiscrete locally comapct group G, $|TLIM(UC(\widehat{G}))| = 2^{2^{b(G)}}$. In particular, we obtain that G is discrete if and only if there is a unique topologically invariant mean on $UC(\widehat{G})$.

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2. Topologically left invariant means on LUC($\mathcal{A}^*$)

Let $\mathcal{A}$ be a Lau algebra. For a linear subspace X of the von Neumann algebra $\mathcal{A}^*$ containing I, the linear functional $m \in X^*$ is called a mean on X if

$$\|m\| = m(I) = 1.$$

For each $a \in \mathcal{A}$, and $f \in \mathcal{A}^*$, define the element $f \cdot a$ in $\mathcal{A}^*$ by

$$(f \cdot a)(b) = f(ab),$$

for all $b \in \mathcal{A}$. A linear subspace X of $\mathcal{A}^*$ is called topologically left invariant if $X \cdot a \subseteq X$ for all $a \in \mathcal{A}$. In this case X is called topologically left introverted if for each $f \in X$ and $n \in \mathcal{A}^{**}$ we have $n \cdot f \in X$, where $n \cdot f$ is defined by

$$(n \cdot f)(a) = n(f \cdot a)$$

for all $a \in \mathcal{A}$. Examples of left introverted subspace of $\mathcal{A}^*$ containing I are $\mathcal{A}^*$ and the space $LUC(\mathcal{A}^*)$ which is the closed linear span in $\mathcal{A}^*$ of $\mathcal{A}^* \cdot \mathcal{A}$. The elements of $LUC(\mathcal{A}^*)$ is called *left uniformly continuous functional*.

Definition 2.1. Let $\mathcal{A}$ be a Lau algebra and let X be a topologically left invariant subspace of $\mathcal{A}^*$ containing I. Then $m \in X^*$ is called a *topologically left invariant mean* (TLIM) on X if $\|m\| = m(I) = 1$ and $\langle m, f \cdot a\rangle = I(a)\langle m, f\rangle$ for all $f \in X$ and $a \in \mathcal{A}$. We denote the set of all topologically left invariant means on X by $TLIM(X^*)$.

Recall that, $\mathcal{A}$ is called *left amenable* if there is a topologically left invariant mean on $\mathcal{A}^*$. Before giving the main result, let us recall that for each $m \in \mathrm{LUC}(\mathcal{A}^*)^*$, we can define a bounded linear map $m_L : \mathcal{A}^* \to \mathcal{A}^*$ as follows

$$m_L(f) = m \cdot f \quad (f \in \mathcal{A}^*).$$

For a Lau algebra $\mathcal{A}$ we denote by $\mathcal{A}_0$ the closed ideal $\{a \in \mathcal{A} : I(a) = 0\}$ and denote by $\langle \mathcal{A}\mathcal{A}_0\rangle$ the closed linear span of $\mathcal{A}\mathcal{A}_0$ in $\mathcal{A}_0$.

Theorem 2.2. *Let $\mathcal{A}$ be a Lau algebra with $\langle \mathcal{A}\mathcal{A}_0\rangle = \mathcal{A}_0$. Then the restriction map $\mathcal{R} : TLIM(\mathcal{A}^{**}) \to TLIM(\mathrm{LUC}(\mathcal{A}^*)^*)$ is a bijection.*

Proof. It is easy to see that $\mathcal{R}$ is well-defined and injective. We need to prove that $\mathcal{R}$ is surjective. Let $n \in TLIM(\mathrm{LUC}(\mathcal{A}^*)^*)$. Define $\widetilde{n} \in \mathcal{A}^{**}$ as follows

$$\widetilde{n}(f) = \langle n_L(f), a_0\rangle \quad (f \in \mathcal{A}^*),$$

where $a_0 \in \mathcal{A}$ with $\|a_0\| = I(a_0) = 1$. It is straightforward to see that

$$\widetilde{n}(I) = n(I) = 1.$$

From this and the fact that $\|a_0\| = 1$, we get that $\|\widetilde{n}\| = 1$. Now, for each $f \in \mathcal{A}^*$ and $a, b \in \mathcal{A}$ we have

$$\langle n_L(f), ab\rangle = \langle n, (f \cdot a) \cdot b\rangle = I(b)\langle n_L(f), a\rangle.$$

Therefore,

$$\langle n_L(f), ab\rangle = 0$$

for all $f \in \mathcal{A}^*$, $a \in \mathcal{A}$ and $b \in \mathcal{A}_0$. It follows from hypothesis that $n_L(f) = 0$ on $\mathcal{A}_0$ for all $f \in \mathcal{A}^*$. In particular, for each $a \in \mathcal{A}$ we have $aa_0 - a_0a \in \mathcal{A}_0$. Thus,

$$\langle \widetilde{n}, f \cdot a\rangle = \langle n_L(f), aa_0\rangle = \langle n_L(f), a_0a\rangle = \langle n, (f \cdot a_0) \cdot a\rangle = I(a)\langle \widetilde{n}, f\rangle.$$

This shows that $\widetilde{n} \in TLIM(\mathcal{A}^{**})$. Moreover, for each $g \in \mathrm{LUC}(\mathcal{A}^*)$,

$$\langle \widetilde{n}, g\rangle = \langle n, g \cdot a_0\rangle = \langle n, g\rangle;$$

that is, $\mathcal{R}(\widetilde{n}) = n$. Hence, $\mathcal{R}$ is surjective. □

Remark 2.3. (i) For the case when $\mathcal{A}$ has a bounded left approximate identity, Lau showed that $\mathcal{A}$ is left amenable if and only if there is a left invariant mean on $\mathrm{LUC}(\mathcal{A}^*)$ [3, Theorem 6.1(b)]. Moreover, it is clear that in this case we have $\langle \mathcal{A}\mathcal{A}_0\rangle = \mathcal{A}_0$. Therefore, Theorem 2.2 is a generalization of [3, Theorem 6.1(b)].

(ii) Examples of Lau algebras satisfying $\langle \mathcal{A}\mathcal{A}_0\rangle = \mathcal{A}_0$ are the group algebra $L^1(G)$ and the Fourier algebra $A(G)$ of a locally compact group G. Moreover, the hypergroup algebra $L^1(H)$ and the measure algebra $M(H)$ of a locally compact hypergroup H with a left Haar measure satisfy this condition. A nice example of Lau algebras satisfying this condition is the quantum group algebra $L^1(\mathbb{G})$. See also [4, 5] for the case $\mathcal{A} = L^1(\mathbb{G})$, where $\mathbb{G}$ is a locally compact quantum group.

In the following results, $|Y|$ stands for the cardinality of a set Y.

Corollary 2.4. *Let $\mathcal{A}$ be a Lau algebra with $\langle \mathcal{A}\mathcal{A}_0\rangle = \mathcal{A}_0$. Then $|TLIM(\mathcal{A}^{**})| = |TLIM(\mathrm{LUC}(\mathcal{A}^*)^*)|$.*

Definition 2.5. Let G be a locally compact group. We denote by $b(G)$ the smallest cardinality of a neighbourhood basis at the identity e for G.

Let G be a locally compact group. Then the Fourier algebra $A(G)$ is a commutative Lau algebra and therefore it is left amenable; see [2]. In the following result we use the following notation $UC(\widehat{G}) := \mathrm{LUC}(A(G))$.

Corollary 2.6. *Let G be a nondiscrete locally comapct group. Then $|TLIM(UC(\widehat{G}))| = 2^{2^{b(G)}}$. In particular, G is discrete if and only if there is a unique topologically invariant mean on $UC(\widehat{G})$.*

Proof. Let $u \in A(G)_0 = \{u \in A(G) : u(e) = 0\}$. Since $\{e\}$ is a set of synthesis for $A(G)$, we can suppose that u has compact support. Using the regularity of $A(G)$, we find $v \in A(G)$ such that $v|_{\mathrm{supp}(u)} = 1$; so that $u = vu \in A(G)A(G)_0$. The rest of the proof follows immediately from Corollary 2.4 and from [1]. □

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