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Analysis of Inpainting via Universal Shearlet Systems

Z. Amiri* and R. A. Kamyabi-Gol

Abstract

Currently, thresholding and compressed sensing in combination with both wavelet and shearlet transforms have been very successful in inpaiting tasks. However, numerical results demonstrate that shearlets outperform wavelets in the problem of image inpainting. In this paper we set up a particular model by inspired seismic data and a box mask to model missing data. The challenge is to fill in the box that is attended in corrupted images.

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1. Introduction

Reconstructing missing data is a popular challenge in both the analog and digital area. Also known as inpainting, this activity is the process of filling in the missing region or techniques for making undetectable modifications to images, modifying the corrupted ones which are not familiar with the original images. Applications of inpainting range from restoring of missing blocks in video data to removal of occlusions such as text from images and repairing of scratched photos. Due to the vast interest in this topic, there exist several excellent reports on inpainting via compressed sensing which is a fundamental method to recover sparsified data by ℓ^1 minimization [3]. The work done in those reports focused on analyzing the concept of clustered sparsity which leads to theoretical bounds and results. Currently, the directional representation systems such as shearlets have been shown to outperform not only wavelets, but also most other directional systems [4]. In addition, superiority of shearlets over wavelets for a basic thresholding algorithm can be found in [3].
In [3], Kutyniok introduced the more flexible universal shearlet systems, which are associated with an arbitrary scaling sequence. We investigate The performance for inpainting of this novel construction shearlet systems.

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2. Abstract Inpainting Framework

We start by analyzing the abstract Hilbert space, which is considered later on. Let x^0 be a signal in separable Hilbert Space $\mathcal{H}$. We assume that $\mathcal{H}$ can be decomposed into a direct sum of two closed subspaces, namely, a subspace $\mathcal{H}_M$ which is associated with the missing part of x^0 and a subspace $\mathcal{H}_K$ which is related to the known part of the signal. Hence, $\mathcal{H} = \mathcal{H}_K \oplus \mathcal{H}_M = P_K\mathcal{H} \oplus P_M\mathcal{H}$,where P_M and P_K denote the orthogonal projections onto those subspaces, respectively. Note that, we will try to find the missing part $P_M x^0$, so the problem of data recovery can be formulated as follows: Given a corrupt signal $P_K x^0$, recover the missing part $P_M x^0$. Depending on the dimension of the given model, we consider $\mathcal{H} = L^2(\mathbb{R}^D)$, $D \in \mathbb{N}$. If the measurable subset $\mathfrak{M} \subseteq \mathbb{R}^D$ is the missing area of the image, we may set $\mathcal{H}_M = L^2(\mathfrak{M})$. Now, we present the methods for recovering a signal which will be useful in the sequel. In fact, one of the fundamental methodologies for spare recovery is ℓ^1 minimization, which recovers the original signal by the following recovery algorithm 1 [3]:

Algorithm 1 Inpainting via ℓ^1 Minimization

Input:
Incomplete signal $P_K x_0 \in \mathcal{H}_K$.
Parseval frame $\Phi = (\phi_i)_{i\in I}$. **Compute:**
(ℓ^1-INP) $\quad x^\star = \text{argmin}_{x\in\mathcal{H}} \|T_\Phi x\|_{\ell^1(I)} \quad$ subject to $P_K x^0 = P_K x$
where T_Φ is analysis operator respect Φ ($T_\Phi : \mathcal{H} \to \ell^2(I), x \to (\langle x, \phi_i\rangle)_{i\in I}$).
Output:
recovered signal $x^\star \in \mathcal{H}$.

Since all Parseval frames are not basis, there are many solutions such c which $x = T^\star_\Phi c$, only the specific solution $T_\Phi x$ produces the desired numerical stabilities. Further, the assumption sparsity signal x^0 by Φ provides a good recovery which is expected to occur.
Now, in order to analyze the optimization problem by inpainting algorithm, we need to introduce two important notions, δ-clustered sparsity and cluster coherence. These notions were applied to study the geometric separation problem and sparsity [1].

Definition 2.1. *[3]. Fix $\delta > 0$. A signal $x \in \mathcal{H}$ is called δ-clustered sparse in a Parseval frame Φ (with respect to $\Lambda \subseteq I$) if*

$$\|1_{\Lambda^c} T_\Phi x\|_{\ell^1} \leq \delta. \tag{1}$$

In this case, Λ is said to be δ-cluster for x in Φ.

The δ-clustered sparsity elucidates that coefficients outside of Λ are small. In fact, the cluster sparsity depends on the chosen set of indices Λ, enlarging Λ leads to smaller δ in (1).
Cluster coherence introduced in [3] to investigate the missing part of signal x^0 on $\mathcal{H}_M$ looks as follows:

Definition 2.2. *[3]. Let $\Lambda \subseteq I$. The cluster coherence $\mu_c(\Lambda, P_M\Phi)$ of Parseval frame Φ with respect to $\mathcal{H}_M$ and Λ is defined by $\mu_c(\Lambda, P_M\Phi) = \max_{j\in J}\sum_{i\in\Lambda} |\langle P_M\phi_i, P_M\phi_j\rangle|$, where $P_M\Phi = (P_M\phi_i)_{i\in I}$.*

In order to clarify the significance of universal shearlet systems, let us recall the main idea of classical shearlet systems . For generator $\psi \in L^2(\mathbb{R}^2)$, a system of shearlet is defined by $\{\psi_{j,l,k} = 2^{\frac{3j}{2}}\psi(S^lA^j[.] - k) : \quad j \in \mathbb{Z}, l \in \mathbb{Z}, k \in \mathbb{Z}^2\}$, where

$$A = \begin{pmatrix} 2^2 & 0 \\ 0 & 2 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

denote the parabolic scaling matrix and shearing matrix, respectively. This approach is enhanced by cone-adapted shearlet systems. The universal shearlet systems were introduced with associated scaling matrix $A^j_{\alpha_j} = \begin{pmatrix} 2^2 & 0 \\ 0 & 2^{\alpha_j} \end{pmatrix}$, where $(\alpha_j)_j \subseteq (-\infty, 2)$ to produce more flexibility in each scale. A sequence $(\alpha_j)_{j\in\mathbb{N}_0} \subseteq \mathbb{R}$ is called a scaling sequence if $\alpha_j \in A_j = \{\frac{m}{j} | m \in \mathbb{Z}, m \leq 2j - 1\} = \{\ldots, \frac{-2}{j}, \frac{-1}{j}, 0, \frac{1}{j}, \frac{2}{j}, \ldots, 2 - \frac{1}{j}\}$, for $j \geq 1$ and $\alpha_0 = 0$.

Definition 2.3. *Let $(\alpha_j)_{j\in\mathbb{N}_0}$ be a scaling sequence. Then universal-scaling shearlet system or universal shearlet system is defined by*

$$\mathrm{SH}(\phi, \nu, (\alpha_j)_j) = \mathrm{SH}_{\mathrm{Low}}(\phi) \cup \mathrm{SH}_{\mathrm{Int}}(\phi, \nu, (\alpha_j)_j) \cup \mathrm{SH}_{\mathrm{Bound}}(\phi, \nu, (\alpha_j)_j).$$

The next Theorem shows that universal shearlet systems are a frame for $L^2(\mathbb{R}^2)$.

Theorem 2.4. *With notations as above, the universal shearlet system is a Parseval frame for $L^2(\mathbb{R}^2)$.*

3. Inpainting with specific Image model on $L^2(\mathbb{R}^2)$

The general approach in this section is the same as in the previous one. We investigate the inpainting results of ℓ^1 minimization by chosen proper index set Λ_j and estimate the relative sparsity and cluster coherence respect to this set.

At the first, we would like to analyze a specific mathematical model which is the model of corrupted line segments. Let $w \in C^\infty(\mathbb{R}^2)$ be a function that is supported in $[-\rho, \rho] \times [-\eta, \eta]$ where $\rho, \eta > 0$. A whole sequence of models $(w_j)_{j\geq 0}$ is given by

$$w_j(x) = w * F_j(x) = \langle w, F_j(x - [\cdot])\rangle, \quad x \in \mathbb{R}^2,$$

where filters F_j are defined by the inverse Fourier transform of the corona functions in [3].

Now, we define the mask of a missing part of image as follows. The mask $\mathcal{M}_h$ is the intersection of a small vertical strip around the x_2-axis and a small horizontal strip around the x_1-axis which is given by $\mathcal{M}_h = \{(x_1, x_2) \in \mathbb{R}^2 : |x_1| \leq h^{x_1}, |x_2| \leq h^{x_2}\}$. For fix some $\varepsilon > 0$, we define the clusters

$$\Lambda_j = \{(j, l, k, \alpha_j, d) : \ |l| < 1, |k_2| < 2^{\varepsilon j}, k \in \mathbb{Z}^2, d = 1, 2\}, \quad j \geq 0.$$

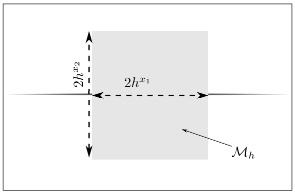

FIGURE 1. Sketch of the corrupted modeling image.

We may determine the relative sparsity of the shearlet coefficient with respect to the cluster $\Lambda_j^{\pm 1}$. Now, we can present the error estimate of Theorem to show the success of image inpainting with special image model. Inpainting result for shearlets and wavelets in special cases can be found in [3].

Theorem 3.1. *Let $(\alpha_j)_j$ be a scaling sequence and $\Psi = \mathrm{SH}(\phi, \upsilon, (\alpha_j)_j)$ be a universal shearlet system. If $\liminf_{j\to\infty} \alpha_j > 0$ and for a fixed $\varepsilon > 0$, $h_j = (h_j^{x_1} \times h_j^{x_2})_j \in o(2^{-(2+\alpha_j+\varepsilon)j})$. Then*

$$\Big(\frac{\|w_j^* - w_j\|_{1,\Psi}}{\|w_j\|_{1,\Psi}}\Big)_j \in o(2^{-Nj}), \quad as \quad j \to \infty$$

for every $N \in \mathbb{N}_0$, where the recover provided by Algorithm 1 is denoted by w_j^*.

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Z. Amiri,
Department of Mathematics, Ferdowsi University of Mashhad
e-mail: za_am10@um.ac.ir

R. A. Kamyabi-Gol ,
Department of Mathematics, Ferdowsi University of Mashhad
e-mail: kamyabi@um.ac.ir