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A note on generalized (σ, τ)-derivations in Banach algebras

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Abstract

Let $\mathcal{A}$ be a Banach algebra , σ and τ are linear mappings (or homomorphisms) on Banach algebra $\mathcal{A}$. A linear map $d : \mathcal{A} \to \mathcal{A}$ is called (σ, τ)-derivation if

$$d(ab) = d(a)\sigma(b) + \tau(a)d(b) \quad a, b \in \mathcal{A}$$

. A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called a (σ, τ)-derivation if there exists a (σ, τ)-derivation $d : \mathcal{A} \to \mathcal{A}$ such that

$$\delta(ab) = \delta(a)\sigma(b) + \tau(a)d(b) \quad (a, b \in \mathcal{A}).$$

In this talk, we investigate automatic continuity of these derivations on Banach algebras.

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1. Introduction

Recently, a number of analysts [2–5] have studied various generalized notions of derivations in the context of Banach algebras. There are some applications in the other fields of research. Such mappings have been extensively studied in pure algebra.

Throughout the paper, $\mathcal{A}$ is always a Banach algebra over complex field $\mathbb{C}$, let σ and τ be two linear mappings (or homomorphisms) on $\mathcal{A}$.A generalized concept of derivation is as follows

Definition 1.1. *A linear mapping $d : \mathcal{A} \longrightarrow \mathcal{A}$ is a (σ, τ)-derivation if $d(ab) = d(a)\sigma(b) + \tau(a)d(b)$ for all $a, b \in \mathcal{A}$, and is a (σ, τ)-inner derivation if there exists $x \in \mathcal{A}$ such that $d(a) = x\sigma(a) - \tau(a)x$ for all $a \in \mathcal{A}$ (see [2] and [5] and references therein).*

Example 1.2. *(i) Every ordinary derivation of an algebra $\mathcal{A}$ into an $\mathcal{A}$-bimodule is an $id_{\mathcal{A}}$-derivation, where $id_{\mathcal{A}}$ is the identity mapping on the algebra $\mathcal{A}$.*

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(ii) Every point derivation $d : \mathcal{A} \to \mathbb{C}$ at the character θ on $\mathcal{A}$ is a θ-derivation.

Definition 1.3. *A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is called generalized (σ, τ)-derivation if there exists a (σ, τ)-derivation $d : \mathcal{A} \to \mathcal{A}$ such that $\delta(ab) = \delta(a)\sigma(b) + \tau(a)d(b) \quad (a, b \in \mathcal{A})$ (see [3] and references therein).*

Example 1.4. *Basic examples are*

(i) (σ, τ)-derivations,

(ii) (σ, τ)-semi inner derivations (i.e., maps of the form $x \rightarrowtail a\sigma(x) + \tau(x)a$ for some $a, b \in \mathcal{A}$) and

(iii) the σ-module maps (i.e., linear maps satisfying $\varphi(ab) = \varphi(a)\sigma(b)$ for all $a, b \in \mathcal{A}$). In particular, if $1 \in \mathcal{A}$, then $\delta(a) = \delta(1)\sigma(a) + d(a)$ for all $a \in \mathcal{A}$ that d is a (σ, id)-derivation.

2. Innerness of generalized (σ, τ)-derivations on Banach algebras

In this section, we introduce generalized (σ, τ)-inner derivation and we get some results about generalized (σ, τ)-derivation.

Definition 2.1. *A linear mapping δ on Banach algebra $\mathcal{A}$ is called generalized (σ, τ)-inner derivation, if there exists σ-module map ψ and $x \in \mathcal{A}$ such that*

$$\delta(a) = \psi(a) - \tau(a)x \quad (a \in \mathcal{A}).$$

Moslehian and Niazi in[5] called a map $d_x : \mathcal{A} \longrightarrow \mathcal{A}$ a (σ, τ)-inner derivation if $d_x(a) = x\sigma(a) - \tau(a)x$ for some $x \in \mathcal{A}$. If we take $\psi(a) = x\sigma(a)$, then this definition covers the notation of Moslehian and Niazi.

Theorem 2.2. *Let δ be a bounded generalized (σ, τ)-derivation on Banach algebra $\mathcal{A}$. Then δ is a generalized (σ, τ)-derivation if and only if there exists an (σ, τ)-inner derivation d_x on $\mathcal{A}$ such that $d_x(a) = x\sigma(a) - \tau(a)x$ and $\delta(ab) = \delta(a)\sigma(b) + \tau(a)d(b) \quad (a, b \in \mathcal{A})$.*

PROOF. Suppose that δ is a generalized (σ, τ)-inner derivation, i.e. there exist σ-module map ψ and $x \in \mathcal{A}$ such that

$$\delta(b) = \psi(b) - \tau(b)x \quad (b \in \mathcal{A}).$$

Now for all $a, b \in \mathcal{A}$ we have

$$\begin{aligned}
\delta(ab) &= \psi(ab) - \tau(ab)x \\
&= \psi(a)\sigma(b) - \tau(a)\tau(b)x \\
&= \Big(\psi(a) - \tau(a)x\Big)\sigma(b) + \tau(a)x\sigma(b) - \tau(a)\tau(b)x \\
&= \Big(\psi(a) = \tau(a)x\Big)\sigma(b) + \tau(a)\Big(x\sigma(b) - \tau(b)x\Big) \\
&= \delta(a)\sigma(b) + \tau(a)d_x(b).
\end{aligned}$$

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conversely, let δ be a generalized (σ,τ)-inner derivation and there exist a inner derivation d_x such that $\delta(ab) = \delta(a)\sigma(b) + \tau(a)d_x(b)$. Define $\psi : \mathcal{A} \longrightarrow \mathcal{A}$ by $\psi(a) = \delta(a) + \tau(a)x$. Then ψ is a bounded linear map. Now we have

$$\begin{aligned}
\psi(ab) &= \delta(ab) + \tau(ab)x \\
&= \delta(a)\sigma(b) + \tau(a)d_x(b) + \tau(a)\tau(b)x \\
&= \delta(a)\sigma(b) + \tau(a)\big(x\sigma(b) - \tau(b)x\big) + \tau(a)\tau(b)x \\
&= \delta(a)\sigma(b) + \tau(a)x\sigma(b) \\
&= \big(\delta(a) + \tau(a)x\big)\sigma(b) \\
&= \tau(a)\sigma(b)
\end{aligned}$$

□

Corollary 2.3. *Let every (σ,τ)-derivation on $\mathcal{A}$ be (σ,τ)-inner derivation. Then every generalized (σ,τ)-derivation on $\mathcal{A}$ is generalized (σ,τ)-inner derivation.*

3. Automatic continuity of generalized (σ,τ)-derivations on Banach algebras

The following proposition characterizes generalized (σ,τ)-derivations by σ-module mappings.

Proposition 3.1. *A linear mapping $\delta : \mathcal{A} \to \mathcal{A}$ is a generalized (σ,τ)-derivation if and only if there exist a (σ,τ)-derivation $d : \mathcal{A} \to \mathcal{A}$ and a σ-module map $\varphi : \mathcal{A} \to \mathcal{A}$ such that $\delta = d + \varphi$.*

Proof. At first suppose δ be a generalized (σ,τ)-derivation on $\mathcal{A}$, then there exists a (σ,τ)-derivation d on $\mathcal{A}$ such that $\delta(ab) = \delta(a)\sigma(b) + \tau(a)d(b) \quad (a, b \in \mathcal{A})$. Therefore $\varphi = \delta - d$ is a σ-module map, because for all $a, b \in \mathcal{A}$ we have

$$\begin{aligned}
\varphi(ab) &= \delta(ab) - d(ab) \\
&= \big(\delta(a)\sigma(b) + \tau(a)d(b)\big) - \big(d(a)\sigma(b) + \tau(a)d(b)\big) \\
&= (\delta(a) - d(a))\sigma(b) \\
&= \varphi(a)\sigma(b).
\end{aligned}$$

Then φ is a σ-module map and $\delta = d + \varphi$.

conversely, let d be a (σ,τ)-derivation on $\mathcal{A}$, φ be a σ-module map on $\mathcal{A}$ and put $\delta = d + \varphi$. Then δ clearly is a linear map and

$$\begin{aligned}
\delta(ab) &= d(ab) + \varphi(ab) \\
&= d(a)\sigma(b) + \tau(a)d(b) + \varphi(a)\sigma(b) \\
&= \big(d(a) + \varphi(a)\big)\sigma(b) + \tau(a)d(b) \\
&= \delta(a)\sigma(b) + \tau(a)d(b)
\end{aligned}$$

for all $a, b \in \mathcal{A}$. Hence δ is a generalized (σ,τ)-derivation. □

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In the following result we investigate continuity of σ-module maps.

Lemma 3.2. *Let $\mathcal{A}$ have a bounded left approximate identity. σ-module map φ on $\mathcal{A}$ is a bounded mapping if σ is a bounded mapping.*

Theorem 3.3. *Let $\mathcal{A}$ have a bounded left approximate identity, σ be bounded linear map on $\mathcal{A}$ and $\delta = d + \varphi$ be a generalized (σ, τ)-derivation. Then δ is bounded if and only if d is bounded.*

Proposition 3.4. *Let σ be a continuous linear mapping on C^*-algebra $\mathcal{A}$. Then every $* - (\sigma, \sigma)$-derivation on $\mathcal{A}$ is automatically continuous.*

Definition 3.5. *Let $\mathcal{A}$ be a $*$-algebra, σ and τ be two linear mappings on $\mathcal{A}$. A linear mapping $\delta : \mathcal{A} \longrightarrow \mathcal{A}$ is called generalized $* - (\sigma, \tau)$-derivations if there exists a $* - (\sigma, \tau)$-derivation $d : \mathcal{A} \longrightarrow \mathcal{A}$ such that*

$$\delta(ab) = \delta(a)\sigma(b) + \tau(a)d(b), \quad (a, b \in \mathcal{A})$$

Theorem 3.6. *Let σ be a continuous linear mapping on C^*-algebra $\mathcal{A}$. Then every generalized $* - (\sigma, \sigma)$-derivation on $\mathcal{A}$ is automatically continuous.*

Proposition 3.7. *If σ and τ are continuous $*$-linear mappings on C^*-algebra $\mathcal{A}$, then every $* - (\sigma, \tau)$-derivation on $\mathcal{A}$ is automatically continuous.*

Theorem 3.8. *If σ and τ are continuous $*$-linear mappings on C^*-algebra $\mathcal{A}$, then every generalized $* - (\sigma, \tau)$-derivation on $\mathcal{A}$ is automatically continuous.*

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