Contractibility of non-Archimedean Banach algebras

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Abstract

In this talk we investigate contractibility of non-Archimedean Banach algebras.

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1. Introduction

Let $\mathbb{K}$ be a field. A non-archimedean absolute value on $\mathbb{K}$ is a function $|.| : \mathbb{K} \longrightarrow [0, \infty)$ such that for any $a, b \in \mathbb{K}$ we have

(i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
(ii) $|ab| = |a||b|$,
(iii) $|a + b| \leq max\{|a|, |b|\}$.

Condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, is concluded from (iii) that $|n| \leq 1$ for each integer n.In all, we always assume that $|.|$ is non-trivial, i.e., that there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin 0, 1$. Let $\mathcal{X}$ be a linear space over a scalar field $\mathbb{K}$ with a non-archimedean non-trivial valuation $|.|$. A function $||.|| : \mathcal{X} \longrightarrow \mathbb{R}$ is a non-archimedean norm (valuation) if it satisfies the following conditions:

(I)$||x|| = 0$ if and only if $x = 0$;
(II) $||rx|| = |r|||x||$ for all $r \in \mathbb{K}$ and $x \in \mathcal{X}$;
(III) the strong triangle inequality (ultrametric); namely, $||x+y|| \leq max\{||x||, ||y||(x, y \in \mathcal{X})$.

Then $(\mathcal{X}, ||.||)$ is called a non-archimedean space. It is concluded from (III) that

$$||x_m - x|| \leq max\{||x_{j+1} - x_j|| : l \leq j \leq m - 1\}(m > l).$$

Therefore, a sequence $\{x_m\}$ is Cauchy in $\mathcal{X}$ if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-archimedean space. By a complete non-archimedean space we mean one in which every Cauchy sequence is convergent. A non-archimedean Banach algebra is a

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complete non-archimedean algebra $\mathcal{A}$ which satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$. For more detailed definitions and history of stability of functions on (normed, Banach, non-archimedean)spaces, we refer to [1].

2. Approximate derivations

Throughout this section, $\mathcal{A}$ is a non-archimedean Banach algebra on non-archimedean filed $\mathbb{K}$ that the characteristic of $\mathbb{K}$ is not 2 and $\mathcal{X}$ is a non-Archimedean Banach $\mathcal{A}$-bimodule.

We say a mapping $f : \mathcal{A} \longrightarrow \mathcal{X}$ is approximately Δ-derivation if it satisfies in a functional equality(inequality)Δ such that there are derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ and real valued function $\varphi : \mathcal{A} \longrightarrow \mathbb{R}$ such that $\|f(a) - D(a)\| \leq \varphi(a)$.

Proposition 2.1. *Suppose that k is a fixed integer greater than 2 and $|k| < |2|$. Let $\mathcal{A}$ be an unital non-archimedean Banach algebra,$\mathcal{X}$ is a non-Archimedean Banach $\mathcal{A}$-bimodule and $f : \mathcal{A} \longrightarrow \mathcal{X}$ be a mapping such that*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\Big(\frac{a+b+cd}{k}\Big)\right\| \tag{1}$$

for all $a, b, c, d, \in \mathcal{A}$. Then f is a derivation.

Proof. By taking $a = b = c = d = 0$ in (1)we have $\|2f(0)\| \leq \|kf(0)\|$. So by $|k| < |2|$ we get $(|2| - |k|)\|f(0)\| \leq 0$ and therefore $f(0) = 0$. Now we show that f is an odd function. Set $b = -a$ and $c = d = 0$ in (1),therefore we have $\|f(a) + f(-a)\| \leq |k|\|f(0)\|$ and so $f(-a) = -f(a)$ for all $a \in \mathcal{A}$. In this step we will show that $f(a^2) = af(a) + f(a)a$ for all $a \in \mathcal{A}$ and therefore we can conclude $f(1) = 0$. For this purpose enough to take $a := a^2, b := 0, c := -a$ and $d := a$ in(1); thus, we get

$$\|f(a^2) + f(0) - af(a) + f(-a)a\| \leq \left\|kf\Big(\frac{a^2+0-aa}{k}\Big)\right\| = 0.$$

Now, letting $c := 1$ and $d := -a - b$ in(1), we have

$$\|f(a) + f(b) - f(a+b)\| \leq \left\|kf\Big(\frac{a+b-(a+b)}{k}\Big)\right\| = 0.$$

As a result, we have $f(a + b) = f(a) + f(b)$. In the last step set $a := ab, b := 0, c : -a$ and $d := b$(1), so we can see that

$$\|f(ab) + f(0) - af(b) - f(a)b\| \leq \left\|kf\Big(\frac{ab+0+a(-b)}{k}\Big)\right\| = 0.$$

Therefore $f(ab) = af(b) + f(a)b$ and this completes the proof. □

Theorem 2.2. *Suppose that k is a fixed integer greater than* 2 *and* $|k| < |2|$. *Let* $r < 1, \theta$ *be nonnegative real numbers and* $f : \mathcal{A} \longrightarrow \mathcal{X}$ *be an odd mapping such that* $f(1) = 0$ *and*

$$(\Delta)\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\left(\frac{a+b+cd}{k}\right)\right\| + \theta(\|a\|^r + \|b\|^r + \|cd\|^r) \tag{2}$$

for all $a, b, c, d \in \mathcal{A}$. *Then there exists a unique derivation* $D : \mathcal{A} \longrightarrow \mathcal{X}$ *such that*

$$\|f(a) - D(a)\| \leq \frac{2 + |2|^r}{|2|^r}\theta\|a\|^r \quad (a \in \mathcal{A}). \tag{3}$$

Theorem 2.3. *Suppose that k is a fixed integer greater than* 2 *and* $|k| < |2|$. *Let* $r > 1, \theta$ *be nonnegative real numbers and* $f : \mathcal{A} \longrightarrow \mathcal{X}$ *be an odd mapping such that* $f(1) = 0$ *and*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\left(\frac{a+b+cd}{k}\right)\right\| + \theta(\|a\|^r + \|b\|^r + \|cd\|^r) \tag{4}$$

for all $a, b, c, d \in \mathcal{A}$. *Then there exists a unique derivation* $D : \mathcal{A} \longrightarrow \mathcal{X}$ *such that*

$$\|f(a) - D(a)\| \leq \frac{2 + |2|^r}{|2|^r}\theta\|a\|^r \quad (a \in \mathcal{A}). \tag{5}$$

Theorem 2.4. *Suppose that k is a fixed integer greater than* 2 *and* $|k| < |2|$. *Let* $r < \frac{1}{3}, \theta$ *be nonnegative real numbers and* $f : \mathcal{A} \longrightarrow \mathcal{X}$ *be an odd mapping such that* $f(1) = 0$ *and*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\left(\frac{a+b+cd}{k}\right)\right\| + \theta\|a\|^r.\|b\|^r.\|cd\|^r) \tag{6}$$

for all $a, b, c, d \in \mathcal{A}$. *Then there exists a unique derivation* $D : \mathcal{A} \longrightarrow \mathcal{X}$ *such that*

$$\|f(a) - D(a)\| \leq \frac{\theta|2|^r}{|2|^{3r}}\|a\|^{3r} \quad (a \in \mathcal{A}). \tag{7}$$

Theorem 2.5. *Suppose that k is a fixed integer greater than* 2 *and* $|k| < |2|$. *Let* $r > \frac{1}{3}, \theta$ *be nonnegative real numbers and* $f : \mathcal{A} \longrightarrow \mathcal{X}$ *be an odd mapping such that* $f(1) = 0$ *and*

$$\|f(a) + f(b) + cf(d) + f(c)d\| \leq \left\|kf\left(\frac{a+b+cd}{k}\right)\right\| + \theta\|a\|^r.\|b\|^r.\|cd\|^r) \tag{8}$$

for all $a, b, c, d \in \mathcal{A}$. *Then there exists a unique derivation* $D : \mathcal{A} \longrightarrow \mathcal{A}$ *such that*

$$\|f(a) - D(a)\| \leq \frac{\theta|2|^r}{|2|}\|a\|^{3r} \quad (a \in \mathcal{A}). \tag{9}$$

3. Approximate Contractible non-Archimedean Banach algebras

If every bounded derivation is inner, then $\mathcal{A}$ is said to be contractible. The non-Archimedean Banach algebra $\mathcal{A}$ is called approximately contractible if for every approximate derivation there exists real valued function $\varphi : \mathcal{A} \longrightarrow \mathbb{R}$ and an $x \in \mathcal{X}$ such that $\|xa - ax - f(a)\| \leq \varphi(a)$

Theorem 3.1. *A non-Archimedean Banach algebra $\mathcal{A}$ is approximately contractible if and only if $\mathcal{A}$ is contractible.*

PROOF. Let $\mathcal{A}$ be a contractible and $f : \mathcal{A} \longrightarrow \mathcal{X}$ is a approximate derivation. By Theorem2.2 there exists a bounded derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ defined by $D(a) := \lim_{n\to\infty} 2^n f\left(\frac{a}{2^n}\right)$, $a \in \mathcal{A}$ which satisfies

$$\|f(a) - D(a)\| \leq \frac{2 + |2|^r}{|2|^r}\theta\|a\|^r \quad (a \in \mathcal{A}). \tag{10}$$

Since $\mathcal{A}$ is contractible, there is some $x \in \mathcal{X}$such that $D(a) = xa - ax$.

Hence $\|xa-ax-f(a)\| = \|D(a)-f(a)\| \leq \frac{2+|2|^r}{|2|^r}\theta\|a\|^r$ Therefore $\mathcal{A}$ is approximately contractible.

Conversely, let $\mathcal{A}$ be approximately contractible and $D : \mathcal{A} \longrightarrow \mathcal{X}$ be a bounded derivation. Then D is trivially an approximate derivation. Due to the approximate contractibility of $\mathcal{A}$, there exists real valued function $\varphi : \mathcal{A} \longrightarrow \mathbb{R}$ and an $x \in \mathcal{X}$ such that $\|xa - ax - D(a)\| \leq \varphi(a)$. Replacing a by $2^n a$ in the later inequality we can conclude $\|xa - ax - D(a)\| \leq 2^{-n}\varphi(a)$. Hence $xa - ax = D(a)$. It follows that $\mathcal{A}$ is contractible. □

One can similarly define notation approximate amenability and establish the following theorem.

Theorem 3.2. *A Non-Archimedean Banach algebra $\mathcal{A}$ is approximately amenable if and only if $\mathcal{A}$ is amenable.*

References

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