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Parseval admissible vectors on hypergroups

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Abstract

In this paper we charactrize Parseval admissible vectors in $L^2(K)$, wherer K is a locally compact hypergroup.

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1. Introduction

Hypergroups, as extensions of locally compact groups, were introduced in a series of papers by C. F. Dunkl, R. I. Jewett and R. Spector . In the last decade, the theory of frame and wavelet analysis has been extended in harmonic analysis on locally compact groups. In some other works we have initiated the concept of admissible vectors on some function spaces related to hypergroups and we have generalized basic properties of coorbit spaces. In this paper, we give a characterization of some special admissible vectors related to the left regular representation of hypergroups and really we extend the main results of [2].

2. Notation and preliminary results

Definition 2.1. *A locally compact Hausdorff space, K, together with a bilinear mapping $(\mu,\nu)\mapsto \mu*\nu$ from $\mathcal{M}(K)\times\mathcal{M}(K)$ into $\mathcal{M}(K)$, and an involutive homeomorphism $x\mapsto x^-$ on K is called a hypergroups if:*

(i) *for each $\mu,\nu\in\mathcal{M}^+(K)$, $\mu*\nu\in\mathcal{M}^+(K)$. Also, the mapping $(\mu,\nu)\mapsto\mu*\nu$ from $\mathcal{M}^+(K)\times\mathcal{M}^+(K)$ into $\mathcal{M}^+(K)$ is continuous, where $\mathcal{M}^+(K)$ is equipped with the cone topology.*

(ii) *$\mathcal{M}(K)$ with $*$ is a complex associative algebra, and for all $\mu,\nu\in\mathcal{M}(K)$ we have*

$$\int_K f d(\mu*\nu)=\int_K\int_K\int_K f d(\delta_x*\delta_y)d\mu(x)d\nu(y),$$

where $f\in C_0(K)$;

* speaker

(iii) *for all* $x, y \in K$, $\delta_x * \delta_y$ *is a compact supported probability measure;*
(iv) *there exists an element* $e \in K$ *(called identity) such that for all* $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$*;*
(v) *for all* $x, y \in K$, $(\delta_x * \delta_y)^- = \delta_{y^-} * \delta_{x^-}$, *where for each* $\mu \in \mathcal{M}(K)$,

$$\mu^-(f) := \int_K f(x^-)d\mu(x), \qquad (f \in C_0(K)).$$

Also, $e \in supp(\delta_x * \delta_y)$ *if and only if* $x = y^-$*;*
(vi) *the mapping* $(x, y) \mapsto supp(\delta_x * \delta_y)$ *from* $K \times K$ *into* $\boldsymbol{C}(K)$ *is continuous, where* $\boldsymbol{C}(K)$ *is the space of all non-empty compact subsets of* K *equipped with Michael topology.*

A hypergroup K is called commutative if for all $x, y \in K$, $\delta_x * \delta_y = \delta_y * \delta_x$. A non-zero non-negative regular measure m on K is called left Haar measure if for each $x \in K$, $\delta_x * m = m$. Any commutative hypergroup has a left Haar measure m.

Definition 2.2. *Let* K *be a commutative hypergroup. A non-zero complex-valued bounded continuous function* ξ *on* K *is called a character if for all* $x, y \in K$, $\xi(x * y) = \xi(x)\xi(y)$ *and* $\xi(x^-) = \overline{\xi(x)}$*. The set of all characters of* K *equipped with the uniform convergence topology on compact subsets of* K*, is denoted by* $\hat{K}$ *and is called the dual of* K*. If* $\hat{K}$ *with the complex conjugation as involution and poinwise product, i.e.*

$$\xi(x)\eta(x) = \int_{\hat{K}} \chi(x)d(\delta_\xi * \delta_\eta)(\chi), \qquad (x \in K \text{ and } \xi, \eta \in \hat{K}),$$

as convolution is a hypergroup, then K *is called a strong hypergroup.*

Definition 2.3. *A complex valued function* f *on* K *is called a trigonometric polynomial if for some* $a_1, \ldots, a_n \in \mathbb{C}$ *and* $\xi_1, \ldots, \xi_n \in \hat{K}$ *we have* $f = \sum_{i=1}^n a_i\xi_i$*. The set of all trigonometric functions on* K *is denoted by* $Trig(K)$*.*

The following well-known theorem is a useful tool in our proofs.

Theorem 2.4. *If* H *is a normal compact subhypergroup of a hypergroup* K*, then* H *has the Weil's property.*

Definition 2.5. *If* H *be a subhypergroup of a hypergroup* K*, then*

$$H^\perp := \{\xi \in \hat{K} : \xi(x) = 1 \text{ for all } x \in H\}$$

is called the annihilator of H *in* $\hat{K}$*.*

Throughout this paper we assume that K is a compact Pontryagin commutative hypergroup and H is a normal compact subhypergroup of K.

3. Main Results

If $f \in L^2(K)$ and $x \in K$, we put $\tau_x f(y) := f(x^- * y)$, where $y \in K$.

Definition 3.1. *For each $\varphi \in L^2(K)$ we denote $A_\varphi :=$ linear span $\{\tau_x\varphi : x \in K\}$, and $\|.\|_2$–closure of A_φ is denoted by V_φ. In this definition, if the elements x are considered from a subhypergroup H of K, then V_φ would be in respect to H.*

Definition 3.2. *Let $\varphi \in L^2(K)$. We denote by $L^2(\hat{H}, w_\varphi)$ the space of all functions $r : \hat{H} \to \mathbb{C}$ with $\int_{\hat{H}} |r(\xi)|^2 w_\varphi(\xi) d\xi < \infty$, where*

$$w_\varphi(\xi) := \int_{H^\perp} |\hat{\varphi}(\xi * \eta)|^2 d\eta.$$

In this case, the mapping

$$\|r\|_\varphi := \left(\int_{\hat{H}} |r(\xi)|^2 w_\varphi(\xi) d\xi \right)^{\frac{1}{2}} \qquad (r \in L^2(\hat{H}, w_\varphi))$$

is a norm on $L^2(\hat{H}, w_\varphi)$,and under above notations, we have $w_\varphi \in L^1(\hat{H})$.

Lemma 3.3. *Let $\varphi \in L^2(K)$. Then $f \in A_\varphi$ if and only if for some $r \in Trig(\hat{K})$, $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi) \quad (\xi \in \hat{K})$.*

Lemma 3.4. *Let $\hat{K}$ is bounded (i.e. there is a constant number $M > 0$ such that for all $\xi \in \hat{K}$, $|\xi| \leq M$), and $\varphi \in L^2(K)$. Then $Trig(\hat{K}) \subseteq L^2(\hat{H}, w_\varphi)$.*

Here we recall the following theorem from [3]

Theorem 3.5. *Let K be a locally compact commutative strong hypergroup with Haar measure m and associated Plancherel measure π, $\epsilon > 0$, and $A := \{\sum_{j=1}^n \lambda_j \langle x_j, \cdot\rangle : n \in \mathbb{N}, \lambda_j \in \mathbb{C}, x_j \in K \ (j = 1, \ldots, n)\}$, where for each $x \in K$ and $\xi \in \hat{K}$, $\langle x, \xi\rangle := \xi(x)$. If $k_1 \in L^2(\hat{K}, \pi)$ has a null zeros set with respect to Plancherel measure π, then for each $k_2 \in L^2(\hat{K}, \pi)$ there exists an element $\psi \in A$ such that $\|k_2 - \psi k_1\|_2 < \epsilon$.*

Corollary 3.6. *Let $\varphi \in L^2(K)$, and $\hat{\varphi} \neq 0$ a.e. on $\hat{K}$. $f \in V_\varphi$ if and only if $\hat{f}(\xi) = r(\xi)\hat{\varphi}(\xi)$ for some $r \in L^2(\hat{H}, w_\varphi)$.*

Definition 3.7. *The mapping $\tau : \mathcal{M}(K) \to B(L^2(K))$ defined by $\tau(\mu)(f) := \mu * f$, where $f \in L^2(K)$ and $\mu \in \mathcal{M}(K)$, is a representation of the hypergroup K called left regular representation.*

Proposition 3.8. *Let $\varphi \in L^2(K)$. The set $\{\tau_x\varphi : x \in K\}$ is an orthogonal system in $L^2(K)$ if $|\hat{\varphi}| = 1$ a.e. on $\hat{K}$.*

Definition 3.9. *Let K be a hypergroup with a (left) Haar measure m, H be a subhypergroup of K, and $\pi : \mathcal{M}(K) \to B(\mathcal{H}_\pi)$ be a representation of K on a Hilbert space $\mathcal{H}_\pi$, and $V \subseteq \mathcal{H}_\pi$. A vector $h_0 \in \mathcal{H}_\pi$ is called a (π, V)–admissible vector with respect to H if there are constant numbers $A, B > 0$ such that for every $h \in V$,*

$$A\|h\|^2 \leq \int_H |\langle \pi_x(h_0), h\rangle|^2 dm_H(x) \leq B\|h\|^2,$$

where m_H is a left Haar measure on H and $\pi_x := \pi(\delta_x)$.

If $A = B = 1$, h_0 is called Parseval admissible.

Definition 3.10. *Let K be a hypergroup. The center of K is defined as*

$$Z(K) := \{x \in K : \delta_x * \delta_{x^-} = \delta_e = \delta_{x^-} * \delta_x\}.$$

Remark 3.11. *In particular, if $H := \{x_k\}_{k=1}^n$ be a subhypergroup of K and $H \subseteq Z(K)$, then $h_0 \in \mathcal{H}_\pi$ is a π–admissible vector with respect to H if and only if there exist constant numbers $A, B > 0$ such that for every $h \in \mathcal{H}_\pi$,*

$$A\|h\|^2 \leq \sum_{k=1}^{n} |\langle \pi_{x_k}(h_0), h\rangle|^2 \leq B\|h\|^2,$$

since in this case m_H is the counting measure.

Theorem 3.12. *A function $\varphi \in L^2(K)$ is a Parseval (τ, V_φ)–admissible vector if and only if $\hat{\varphi} = \chi_{\Omega_\varphi}$ a.e. on $\hat{K}$, where $\Omega_\varphi := supp(\hat{\varphi})$.*

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