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# Hecke $*$-Algebras on Hypergroups

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**Abstract**

In this paper we initiate and study the Hecke $*$-algebra for a discrete hypergroup $K$ and its $m$–subhypergroup $H$.

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## 1. Introduction and Preliminaries

In this work we extend the ideas in [5] and [2] for a hypergroup pair $(K, H)$ in which $K$ is a discrete hypergroup and $H$ is an $m$-subhypergroup of $K$, where $m$ is a left Haar measure of $K$. This new structure coincides classical Hecke algebra in the group case. We show that a property putting on elements of $\Omega$, called condition $(\beta)$, is equivalent with the associativity of the convolution product between elements of $\Omega$. Our main references for hypergroups are [1], [3] and [4].

Definition 1.1. *Suppose that $K$ is a locally compact Hausdorff space, $(\mu, \nu) \mapsto \mu * \nu$ is a bilinear positive-continuous mapping from $M(K) \times M(K)$ into $M(K)$ (called convolution), and $x \mapsto x^-$ is an involutive homeomorphism on $K$ (called involution) with the following properties:*

(i) *$M(K)$ with $*$ is a complex associative algebra;*
(ii) *if $x, y \in K$, then $\delta_x * \delta_y$ is a probability measure with compact support;*
(iii) *the mapping $(x, y) \mapsto supp(\delta_x * \delta_y)$ from $K \times K$ into $\boldsymbol{C}(K)$ is continuous, where $\boldsymbol{C}(K)$ is the set of all non-empty compact subsets of $K$ equipped with Michael topology;*
(iv) *there exists a (necessarily unique) element $e \in K$ (called identity) such that for all $x \in K$, $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$;*
(v) *for all $x, y \in K$, $e \in supp(\delta_x * \delta_y)$ if and only if $x = y^-$;*

*Then $K \equiv (K, *, ^-, e)$ is called a locally compact hypergroup.*

* speaker

Let $f$ and $g$ be complex-valued Borel measurable functions on $K$. For each $x, y, z \in K$ we define

$$f_x(y) = f^y(x) = f(x * y) := \int_K f \, d(\delta_x * \delta_y) \quad \text{and} \quad f(x * y * z) := f_x(y * z).$$

Throughout this paper, we assume that $K$ is a discrete hypergroup with left Haar measure $m$, and $H$ is a subhypergroup of $K$. We denote by $C(K)$ the space of all finite supported complex-valued functions on $K$. We denote $H \backslash K := \{H * a : a \in K\}$, $K/H := \{a * H : a \in K\}$, and $H \backslash K/H := \{H * a * H : a \in K\}$. The set of all functions $f \in C(K)$ such that $f$ is constant on all left and right cosets of $H$ is denoted by $\mathrm{CT}(H)$.

## 2. Main Results

Definition 2.1. *Let $K$ be a discrete hypergroup with a Haar measure $m$, and $H$ be a subhypergroup of $K$. We say that $H$ satisfies condition ($\beta$) if for every $f \in CT(H)$, every set $A$ of representatives of distinct elements of $H \backslash K$ and each $a \in K$, there is a set $B$ of representatives of distinct elements of $H \backslash K$ such that $(\chi_A f)^a \, dm = \chi_B f^a \, dm$, i.e. for all $g \in C(K)$,*

$$\int_K g(x)(\chi_A f)(x * a) \, dm(x) = \int_B g(x) f(x * a) \, dm(x).$$

Proposition 2.2. *Let $G$ be a discrete group and $H$ be a subgroup of $G$. Then $H$ satisfies in condition ($\beta$).*

Definition 2.3. *Let $K$ be a discrete hypergroup and $H$ be a subhypergroup of $K$ satisfying the condition ($\beta$). The set of all functions $f \in C(K)$ satisfying following properties is denoted by $\Omega$:*

1. *for each $x \in K$, $f$ is constant on $H * x$;*
2. *for each $x \in K$, $f$ is constant on $x * H$;*
3. *for each $x, y, z \in K$, $h \in H$ and $\mu, \nu \in M_+^c(K)$, if $x * y \subseteq x * z * h$, then*

$$\int_K f \, d(\mu * \delta_x * \delta_y * \nu) = \int_K f \, d(\mu * \delta_x * \delta_z * \delta_h * \nu);$$

4. *for each $x, y, z \in K$, $h \in H$ and $\mu, \nu \in M_+^c(K)$, if $y * x \subseteq h * z * x$, then*

$$\int_K f \, d(\mu * \delta_y * \delta_x * \nu) = \int_K f \, d(\mu * \delta_h * \delta_z * \delta_x * \nu).$$

Definition 2.4. *Let $K$ be a discrete hypergroup and $m$ be a left Haar measure of $K$. A subhypergroup $H$ of $K$ is called $m$-subhypergroup if $\chi_A dm = \chi_B dm$, where $A$ and $B$ are two arbitrary sets of representatives of distinct elements of $H \backslash K$.*

Definition 2.5. *Let $K$ be a discrete hypergroup with a left Haar measure $m$, and $H$ be an $m$–subhypergroup of $K$. For each $f, g \in \Omega$ and $x \in K$, we define*

$$f * g(x) := \int_A f(x * y^-) g(y) \, dm(y) \qquad \textit{and} \qquad f^*(x) := \overline{f(x^-)},$$

*where in the above integral, $A$ is a set of representatives of distinct elements of $H\backslash K$. In Theorem 2.8 we will show that the above convolution does not depend on the choice of the set $A$.*

Lemma 2.6. *Let $K$ be a discrete hypergroup with a left Haar measure $m$, $H$ be a subhypergroup of $K$ satisfying the condition $(\beta)$, $f, g \in C(K)$, and $a \in K$. If $A$ is a set of representatives of distinct elements of $H\backslash K$, then there is a set $B$ of representatives of distinct elements of $H\backslash K$, such that*

$$[\chi_A (f * g)]^a = [(\chi_B f) * g]^a.$$

Proposition 2.7. *Let $K$ be a discrete hypergroup with a left Haar measure $m$, $H$ be a subhypergroup of $K$ satisfying the condition $(\beta)$, $f, g \in C(K)$, and $a \in K$. For every set $B$ of representatives of distinct elements of $H\backslash K$, there is a set $E$ of representatives of distinct elements of $H\backslash K$ such that*

$$\int_K h(y)[\chi_A (f * g)](y * a)\, dm(y) = \int_K (\chi_E h^{a^-})(y)(f * g)(y) dm(y).$$

Theorem 2.8. *Let $K$ be a discrete hypergroup with a left Haar measure $m$, $H$ be an $m$-subhypergroup of $K$ satisfying the condition $(\beta)$. Then with convolution and involution introduced in Definition 2.5, the space $\Omega$ is an $*$-algebra.*

Corollary 2.9. *Let $K$ be a discrete hypergroup and $H$ be a subhypergroup of $K$. $\Omega$ with the convolution and involution introduced in Definition 2.5 is a $*$–algebra if and only if $H$ satisfy in the condition $(\beta)$.*

Definition 2.10. *The set $\Omega$ with above convolution and involution is denoted by $H(K, H)$ and is called hyper-Hecke $*$-algebra.*

Remark 2.11. *Note that in the case that $G$ is a discrete group, $\Gamma$ is a subgroup of $G$, $K := \Gamma\backslash G/\Gamma$, doubel cosets of $\Gamma$ in $G$, and $H := \{\Gamma e\Gamma\}$, the trivial subhypergroup of $K$, then $H(K, H)$ is the classical $*$-Hecke algebra.*

Theorem 2.12. *If $K$ is a discrete hypergroup and $H$ is a subhypergroup of $K$, then $H$ satisfies the condition $(\beta)$ if and only if $K$ is a group. In the other words, the space $\Omega$ related to $(K, H)$ is an $*$–algebra if and only if $K$ is a group.*

## References

[1] W. R. Bloom and H. Heyer, Harmonic Analysis of Probability Measures on Hypergroups, De Gruyter, Berlin, 1995.

[2] J.-B. Bost and A. Connes, Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory, *Selecta Math. (N. S.)*, **1**(1995), 411-457.

[3] S. M. Tabatabaie, The problem of density on $L^2(G)$, *Acta Math. Hungarica*, **150** (2016), 339-345.

[4] S. M. Tabatabaie, The problem of density on commutative strong hypergroup, *to appear in Math. Reports*

[5] K. Tzanev, Hecke $C^*$-algebras and amenability, *J. Operator Theory*, **50**(2003), 169-178.

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