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Sampling and Reproducing Kernel Hilbert Space

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Abstract

Reproducing kernel Hilbert spaces arise in a number of areas, including approximation theorys statistics, machine learning theory, group representation theory and various areas of complex analysis. In this talk the reproducing kernel Hilbert spaces are introduced and their general properties are investigated. For better understanding of these spaces some examples are given. Moreover, in this space, a sampling formula is introduced, which is an extension of a famous formula known as the Paley-Wiener (PW) of band-limited signals. In particular, a single channel sampling formula is discussed.

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1. Introduction

The theory of Hilbert spaces of entire functions was first introduced by L. de Branges in the series of papers. These spaces, which are now called de Branges spaces, generalize the classical Paley-Wiener space which consists of the entire functions of exponential type which are square integrable on the real line. Recently, Ortega Cerd'a and K. Seip provided a description of the exponential frames for the Paley-Wiener space, and a related study of sampling and interpolation, by connecting the problem to the de Branges spaces theory of entire functions.
Let f be a band-limited signal with band region $[-\pi, \pi]$, that is, a square integrable function on $\mathbb{R}$ of which the Fourier transform $\hat{f}$ vanishes outside $[-\pi, \pi]$.Then f can be recovered by its uniformly spaced discrete values as $f(t) = \sum_{-\infty}^{\infty} f(n)\dfrac{sin\pi(t-n)}{\pi(t-n)}$, which converges absolutely and uniformly over $\mathbb{R}$. This series is called the cardinal series or the Whittaker-Shannon-Kotelnikov (WSK) sampling series. This formula tells us that once we know the values of a band-limited signal f at certain discrete points, we can recover f completely. In1941, Hardy [2] recognized that this cardinal series is actually an orthogonal expansion. WSK sampling series was generalized by Kramer [4] in 1957 as follows: Let $k(\xi, t)$ be a kernel on $I \times X$, where I is a bounded interval and X is a subset of $\mathbb{R}$. Assume that $k(., t) \in L^2(I)$ for each t in X and there

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are points $\{t_n\}_{n\in\mathbb{Z}}$ in X such that $k(\xi, t_n)$ is an orthonormal basis of $L^2(I)$. Then any $f(t) = \int_I F(\xi)k(\xi, t)d\xi$ with $F(\xi) \in L^2(I)$ can be expressed as a sampling series.

$$f(t) = \sum_n f(t_n) \int_I k(\xi, t)\overline{k(\xi, t_n)}d\xi,$$

which converges absolutely and uniformly over the subset $\mathbb{D}$ on which $\|k(., t)\|_{L^2(I)}$ is bounded. While WSK sampling series treats sample values taken at uniformly spaced points, Kramers series may take sample values at nonuniformly spaced points.
Recently, A. G. Garcia and A. Portal [1] extended the WSK and Kramer sampling formulas further to a more general setting using a suitable abstract Hilbert space valued kernel. On the other hand, Papoulis introduced a multi-channel sampling formula for band-limited signals, in which a signal is recovered from discrete sample values of several transformed versions of the signal.

2. Preliminaries and notations

We will consider Hilbert spaces over either the field of real numbers, $\mathbb{R}$, or of complex numbers, $\mathbb{C}$. We will use $\mathbb{F}$ to denote either $\mathbb{R}$ or $\mathbb{C}$, so that when we wish to state a definition or result that is true for either the real or complex numbers, we will use $\mathbb{F}$. Given a set X, if we equip the set of all functions from X to $\mathbb{F}$, $\mathcal{F}(X, \mathbb{F})$ with the usual operations of addition, $(f + g)(x) = f(x) + g(x)$, and scalar multiplication, $(\lambda.f)(x) = \lambda.(f(x))$, then $\mathcal{F}(X, \mathbb{F})$ is a vector space over $\mathbb{F}$.

Definition 2.1. *Given a set X, we will say that $\mathcal{H}$ is a reproducing kernel Hilbert space (RKHS) on X over $\mathbb{F}$, provided that:*

(i) *$\mathcal{H}$ is a vector subspace of $\mathcal{F}(X, \mathbb{F})$,*
(ii) *$\mathcal{H}$ is endowed with an inner product, $<, >$, making it into a Hilbert space,*
(iii) *for every $y \in X$, the linear evaluation functional, $E_y : \mathcal{H} \longrightarrow \mathbb{F}$, defined by $E_y(f) = f(y)$, is bounded*

If $\mathcal{H}$ is a RKHS on X, then since every bounded linear functional is given by the inner product with a unique vector in $\mathcal{H}$, we have that for every $y \in X$, there exists a unique vector, $k_y \in \mathcal{H}$, such that for every $f \in \mathcal{H}$, $f(y) =< f, k_y >$.

Definition 2.2. *The function k_y is called the reproducing kernel for the point y. The 2-variable function dened by $K(x, y) = k_y(x)$ is called the reproducing kernel for $\mathcal{H}$. Note that we have, $K(x, y) = k_y(x) =< k_y, k_x >$ and $\|E_y\|^2 = \|k_y\|^2 =< k_y, k_y >= K(y, y)$.*

We now look at a few key examples
The Hardy Space of the Unit Disk $H^2(\mathbb{D})$:
$H^2(\mathbb{D}) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n : \quad \{a_n\} \in \ell^2(\mathbb{N} \cup \{0\}), z \in \mathbb{D}\}$, with the inner product, $< f, g >_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n\overline{b_n}$, and the reproducing kernel $K(z, w) = \dfrac{1}{1 - \overline{w}z}$. This function is called the **Szego kernel** on the disk.
Sobolev Spaces on [0,1]:

$\mathcal{H} = \{f : [0,1] \longrightarrow \mathbb{R}, f' \in L^2[0,1], f(0) = f(1) = 0,$ f is absolutely continuous$\}$. with the inner product, $< f, g >= \int_0^1 f'(t)g'(t)dt$, and the reproducing kernel

$$k(x,y) = k_y(x) = \begin{cases} (1-y)x, & x \le y \\ (1-x)y, & x \ge y \end{cases} .$$

3. Main results

Let X be any set and let $\mathcal{H}$ be a RKHS on X with kernel K. In this section, we will show that K completely determines the space $\mathcal{H}$ and characterize the functions that are the kernel functions of some RKHS.

Proposition 3.1. *Let $\mathcal{H}$ be a RKHS on the set X with kernel K . Then the linear span of the functions, $k_y(.) = K(., y)$ is dense in $\mathcal{H}$.*

Lemma 3.2. *Let $\mathcal{H}$ be a RKHS on X and let $\{f_n\} \subseteq \mathcal{H}$. If $\lim_n \|f_n - f\| = 0$, then $f(x) = \lim_n f_n(x)$ for every $x \in X$.*

Proposition 3.3. *Let $\mathcal{H}_i$, $i = 1,2$ be RKHSs on X with kernels, $K_i(x,y)$, $i = 1,2$. If $K_1(x,y) = K_2(x,y)$ for all $x, y \in X$, then $H_1 = H_2$ and $\|f\|_1 = \|f\|_2$ for every f.*

Theorem 3.4. *Let $\mathcal{H}$ be a RKHS on X with reproducing kernel, $K(x,y)$. If $\{e_s : s \in S\}$ is an orthonormal basis for $\mathcal{H}$, then $K(x,y) = \sum_{s\in S} \overline{e_s(y)} e_s(x)$ where this series converges pointwise.*

Theorem 3.5. *Let $\mathcal{H}$ be a RKHS on X with reproducing kernel $K(x,y)$. Then $\{f_s : s \in S\} \subseteq \mathcal{H}$ is a Parseval frame for $\mathcal{H}$ if and only if $K(x,y) = \sum_{s\in S} f_s(x)\overline{f_s(y)}$, where the series converges pointwise.*

Let $\mathcal{H}$ be a separable Hilbert space and $k : X \longrightarrow \mathcal{H}$ be an H-valued function on a subset X of the real line $\mathbb{R}$. Define a linear operator T on $\mathcal{H}$ by

$$T(x)(t) = f_x(t) :=< x; k(t) >_{\mathcal{H}}; t \in X$$

We call $k(t)$ the kernel of the linear operator T.

Lemma 3.6. (a) *T is one-to-one if and only if $\{k(t)|t \in X\}$ is total in $\mathcal{H}$.*

(b) *Assume $\{k(t)|t \in X\}$ is total in $\mathcal{H}$ so that $T : \mathcal{H} \longrightarrow T(\mathcal{H})$ is a bijection. Then $< T(x), T(y) >_{T(\mathcal{H})} :=< x, y >_{\mathcal{H}}$ defines an inner product on $T(\mathcal{H})$, with which $T(\mathcal{H})$ is a Hilbert space and $T : \mathcal{H} \longrightarrow T(\mathcal{H})$ is unitary. Moreover, $T(\mathcal{H})$ becomes an RKHS with the reproducing kernel $k(s,t) :=< k(t), k(s) >_{\mathcal{H}}$.*

Theorem 3.7. *If $\ker T \subseteq \ker \tilde{T}$ and there exists a sequence $\{t_n\}_{n=1}^{\infty}$ in X such that $\{\tilde{k}(t_n)\}_n$ is a basis of $\mathcal{H}$, then T is one-to-one so that $T(\mathcal{H})$ becomes an RKHS under the inner product $< x, y >_{\mathcal{H}}=< T(x), T(y) >_{T(\mathcal{H})}$. Moreover, there is a basis $\{S_n\}_{n=1}^{\infty}$ of $T(\mathcal{H})$ with which we have the sampling expansion, $f_x(t) = \sum_{n=1}^{\infty} \tilde{f}_x(t_n) S_n(t)$, $(f_x(.) \in T(\mathcal{H}))$ which converges not only in $T(\mathcal{H})$ but also uniformly over any subset on which $\|k(t)\|$ is bounded.*

Theorem 3.8. ***(Asymmetric nonuniform multi-channel sampling formula)***
If $\ker T \subseteq \bigcap_{i=1}^{N} \ker T_i$ *and there exist points* $\{t_{i,n}, 1 \leq i \leq N, n \in \mathbb{Z}\} \subset X$ *and constants* $\{\alpha_{i,n}^{j}, 1 \leq i \leq N, 1 \leq j \leq M, n \in \mathbb{Z}\}$ *for some integer* $M \geqslant 1$ *such that* $\{\sum_{i,n}^{N} \alpha_{i,n}^{j} k_i(t_{i,n}) : 1 \leq j \leq M, n \in \mathbb{Z}\}$ *is an unconditional basis of* $\mathcal{H}$*, then there is a basis*$\{S_{j,n}(t) : 1 \leq j \leq M, n \in \mathbb{Z}\}$ *of* $T(\mathcal{H})$ *such that for any* $f_x(t) = T(x)(t) \in T(\mathcal{H})$*,*

$$f_x(t) = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{m} \{\overline{\alpha_{1,n}^{j}} f_x^1(t_{1,n}) + \overline{\alpha_{2,n}^{j}} f_x^2(t_{2,n}) + \ldots + \overline{\alpha_{N,n}^{j}} f_x^N(t_{N,n})\} S_{j,n}(t) \tag{1}$$

which converges in $T(\mathcal{H})$*. Moreover, the series (1) converges absolutely and uniformly on any subset of* X *over which* $\|k(t)\|_{\mathcal{H}}$ *is bounded.*

4. Examples

(Sampling with Hilbert transform): Take $\tilde{k}(t)(\xi) = i\ sgn(\xi)(k(t))(\xi)$ so that $\tilde{T}(f)(t) = \tilde{f}_x(t)$ is the Hilbert transform of $f(t)$ in PW_π . Choosing $\{t_n\}_{n\in\mathbb{Z}} = \{n\}_{n\in\mathbb{Z}}$, $\{x_n\}_{n\in\mathbb{Z}} = \{i\ sgn\xi \frac{e^{-in\xi}}{\sqrt{2\pi}}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $L^2[-\pi,\pi]$ so that $\{x_n^*\}_{n\in\mathbb{Z}} = \{x_n\}_{n\in\mathbb{Z}}$
We then have

$$S_n(t) = T(x_n^*)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} i\ sgn(\xi) \frac{e^{-in\xi}}{\sqrt{2\pi}} e^{it\xi} d\xi = -sinc\frac{1}{2}(t-n) sin\frac{\pi}{2}(t-n)$$

where $sinct := \frac{sin\pi t}{\pi t}$, Hence, we have

$$f_x(t) = -\sum_{n\in\mathbb{Z}} \tilde{f}(n) sinc\frac{1}{2}(t-n) sin\frac{\pi}{2}(t-n), \quad (f \in PW_\pi)$$

Using the operational relation $\tilde{\tilde{f}}(t) = -f(t)$ and the fact that if$f \in PW_\pi$, then so does $\tilde{f} \in PW_\pi$, we also have $\tilde{f}(t) = \sum_{n\in\mathbb{Z}} f(n) sinc\frac{1}{2}(t-n) sin\frac{\pi}{2}(t-n), (f \in PW_\pi)$

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