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Wavelet on 2-sphere

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Abstract

in this paper, we extends the continuos wavelet transform on sphere, which the foundation of building wavelet on each manifold is based on group theoretical approach. In this technique, wavelets are considered as an ordered positions related to group theoretical.

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1. Introduction

Analyzing data with the continuous wavelet transform (CWT) is by now a wellestablished procedure [1]. The most common cases are data on the line (signal processing), on the plane (image analysis), or occasionally in R^3 (e.g., in fluid dynamics) for a survey of applications in physics. However, there are various instances where data are given on a sphere. Geophysical data are the prime example, but others occur in statistical problems, computer vision, or medical imaging. The problem is to adapt the method of analysis to spherical data. Of course, this is not specific to wavelet analysis, but shows up in all methods, mostly based on Fourier techniques, and it is in general a nontrivial task from the numerical point of view. So the question arises, how does one extend the CWT to the sphere or a manifold? In order to obtain a genuine CWT on S^2, the following three requirements should be satisfied:
∗ the signals and the wavelets must live on the sphere;
∗ the transform must involve (local) dilations of some kind; and
∗ possibly the CWT on S^2 should reduce locally to the usual CWT on the (tangent) plane (Euclidean limit).
This paper starts with sketch out the metod of construction of wavelets on any manifold, based on group theory. Next, the construction of continuous wavelet transform (CWT) on the 2-sphere, S^2, based on the construction of general coherent states associated to square integrable group representations. The parameter space X of our CWT is the product of SO(3) for motions and R^+_* for dilations on S^2, which are embedded in

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to the Lorentz group $SO_0(3,1)$ via the Iwasawa decomposition, so that $X \cong \frac{SO_0(3,1)}{N}$, where $N \cong C$. We select an appropriate unitary representation of $SO_0(3,1)$ acting in the space $L^2(S^2, d\mu)$ of finite energy signals on S^2 . This representation is square integrable over X, thus it yields immediately the wavelets on S^2 and the associated CWT. Finaly, it is given efficently application of wavelet transform on sphere. including of noise detectation in medical images such as, brain. [2]

2. Preliminaries and notations

Definition 2.1. *A manifold M of dimension n, or n- manifold , is a topological space with the following properties:*

1. *M is Hausdorff,*
2. *M is locally Euclidean of dimension n, and*
3. *M has a countable basis of open sets.*

3. Second section

consider the space of finite energy signals $\mathcal{H} = L^2(S^2, d\mu)$, where $d\mu = sin\theta d\theta d\varphi$ is the usual (rotation invariant) measure on S^2. The first step for constructing a CWT on S^2 is to identify the appropriate transformations. These are of two types, displacements, also called motions, and dilations:
1. Motions are given by elements of the rotation group $SO(3)$, which indeed acts transitively on S^2, and $S^2 \cong \frac{SO(3)}{SO(2)}$
2. Dilations may be derived in two steps:
• dilations around the North Pole are obtained by considering usual dilations in the tangent plane at the North Pole and lifting them to S^2 by inverse stereographic projection from the South Pole;
• a dilation around any other point $\omega \in S^2$, is obtained by moving ω to the North Pole by a rotation $\gamma \in SO(3)$, performing a dilation D_N as before and going back by the inverse rotation:

we will not deal with the full Lorentz group, since we are only interested in the action of dilations and motions. We thus restrict ourselves to the corresponding homogeneous space using a suitable section $\sigma : X = \frac{KAN}{N} \to KAN$ in the principal fiber bundle defined by the Iwasawa decomposition. Thus we will concentrate on the reduced expression

$$[U(\sigma(x))f](\omega) = \lambda(\sigma(x), \omega)^{\frac{1}{2}} f(\sigma(x)^{-1}\omega), \tag{1}$$

We write points of the space X as pairs $x \equiv (\gamma, a)$, with $\gamma \in SO(3)$ and $a \in A \simeq SO(1,1)$, and choose the natural (Iwasawa) section $\sigma_1(\gamma, a) = \gamma a$. We have already computed the action of dilations and, by $SO(3)$ invariance, it is easily seen that, with $\omega = (\theta, \varphi)$, $\lambda(\sigma_1(\gamma, a), \omega) \equiv \lambda(a, \theta) = \frac{4a^2}{[(a^2-1)cos\theta + (a^2+1)]^2}$.

we will build in this section a system of coherent states for the Lorentz group, indexed by points of the homogeneous space $X = \frac{SO_0(3,1)}{N}$. The CS system associated to the representation U on (1) is defined by $\eta_{\sigma(x)}(\omega) = [U(\sigma(x))\eta](\omega)$, with $\eta \in L^2(S^2, d\mu)$ and $x \in X$, that is, the elements of the orbit of η under G, modulo the section σ . if the representation U is square integrable modulo the section σ and the subgroup N. This means that there exists a nonzero vector $\eta \in L^2(S^2, d\mu)$, called admissible, such that

$$\int_X |\langle U(\sigma(x))\eta|\phi\rangle|^2 < \infty \qquad \forall \phi \in L^2(S^2, d\mu).$$

In this expression, ν is an $SO_0(3,1)$-invariant measure on X, and the inner product in the integrand is taken in $L^2(S^2, d\mu)$.

Theorem 3.1. *A function $\eta \in L^2(S^2, d\mu)$ is admissible only if it satisfies the condition* $\int_{S^2} \frac{\eta(\theta,\varphi)}{1+cos\theta} d\mu(\theta,\varphi) = 0$.

Definition 3.2. *Consider admissible wavelet $\psi \in L^2(S^2, d\mu)$, wavelets on sphere are $\psi_{\gamma,a} = U(\sigma(\gamma, a))\gamma$ and countinus wavelet transform of a signal defined as:*

$$\begin{aligned} W_f(\rho, a) &= \langle f, \psi_{\rho,a}\rangle \\ &= \int_{S^2} \overline{R_\rho D_a \psi(\omega)} f(\omega) d\mu(\omega) \\ &= \int_{S^2} \overline{\psi_a(\rho^{-1}\omega)} f(\omega) d\mu(\omega) \end{aligned}$$

4. Third Section

Subtle changes in human cortical thickness are thought to be associated with neurological or clinical deficiencies. It is therefore important to detect these changes using bioimagingmodalities. Typically, cortical thicknessmaps are inferred over the brain surface through magnetic resonance imaging (MRI) examinations. Cortical thickness features are extracted from these maps and studied using statistical tests. The acquisition process is, however, unavoidably affected by noise. As discussed in [2], the outcome of the statistical tests is known to be greatly improved by spatial smoothing, specifically with low-pass Gaussian filters, which enhances the signal-to-noise ratio. Since the cortex is a very convoluted surface, there is no easy way directly to apply simple low-pass filtering. In this context, scale-discretized wavelets on the sphere offer a very flexible and computationally efficient way to denoise data while preserving the most salient features and most important spatial variations. Denoising with wavelets obviously require a discrete formalism where the signal may be reconstructed after modification of its wavelet coefficients. In [2], the spherical cortical thickness map is processed using scale-discretized axisymmetric wavelets on the sphere. The wavelet coefficients are then thresholded by soft thresholding in order to remove the noise, which is assumed to be uniformly distributed over the coefficients, while the most important spatial variations of the original data are encoded in the strongestwavelet coefficients only. This study shows that wavelet denoising yields a significant improvement over spatial smoothing by Gaussian filtering.

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References

[1] J. P. Antoine and P.Vandergheynst, *Wavelets on the 2-Sphere: a group Theoretical Approach*. Appl. Comput. Harmon. Anal., 7(1999), 262-291.

[2] J. L. Bernal-Rusiel, M. Atienza and J. L. Cantero, *Detection of focal changes in human cortical thickness: Spherical Wavelets versus Gaussian smoothing*, NeuroImage, 41(2008), 1278-1292.

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