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Submitted to the 5st Seminar on Harmonic Analysis and Applications
Organized by the Iranian Mathematical Society
January 18–19, 2017, Ferdowsi University of Mashhad, Iran

پنجمین سمینار آنالیز هارمونیک و کاربردها
5th Seminar on Harmonic Analysis and Applications

The condition of uniqueness dual for frames and K-frames

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Abstract

Frames in Hilbert spaces are a redundant of vectores which yield a representation for each vector in the space. K-frames were recently introduced by Găvruţa in Hilbert spaces. They respect to a bounded linear operator K. In this paper, we will present some conditions of uniqueness dual frame and dual K-frame in cases.

2010 *Mathematics subject classification:* Primary 42C15; Secondary 41A58.
Keywords and phrases: Frame, K-frame, Dual K-frame, K-minimal frame, Uniqueness.

1. Introduction

Frames in Hilbert spaces were introduced by J. Duffin and A.C. Schaffer in 1952. One of the essential applications of frames is that they provide basis-like but generally nonunique decompositions, dual frames play a key role. Găvruţa recently presented a generalization of frames with a linear bounded operator K. There are many dual for frames and K-frames, in case the canonical dual frame. But we can find conditions that make this existance in a unique way.
First introduce basic definitions as follow:

Definition 1.1. *A Riesz basis for H is a family of the form $\{Ue_n\}_{n=1}^{\infty}$, where $\{e_n\}_{n=1}^{\infty}$ is an orthonoamal basis for H and $U : H \to H$ is a bounded bijective operator.*

Definition 1.2. *A family of elements $\{f_n\}_{n=1}^{\infty} \subset H$ is called a frame of H if there exist constant $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n\rangle|^2 \leq B\|f\|^2, \qquad \forall f \in H.$$

The constant A, B are called frame bound. The sequence $\{f_n\}_{n=1}^{\infty}$ is said to be Bessel sequence for H if we only require the right-hand inequality.

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Definition 1.3. *A Bessel sequence* $\{g_n\}_{n=1}^{\infty}$ *in* H *is called a dual frame for the frame* $\{f_n\}_{n=1}^{\infty}$ *if*

$$f = \sum_{n=1}^{\infty} \langle f, g_n \rangle f_n, \qquad \forall f \in H.$$

Definition 1.4. *A sequence* $\{f_n\}_{n=1}^{\infty} \subset H$ *is called a* K*-frame for* H *if there exist two constants* $0 < A \leq B < \infty$ *such that*

$$A\|K^* f\|^2 \leq \sum_{n=1}^{\infty} |\langle f, f_n \rangle|^2 \leq B\|f\|^2, \qquad \forall f \in H.$$

The numbers A, B *are called* K*-frame bounds.*

Definition 1.5. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a* K*-frame for* H*. We call a Bessel sequence* $\{g_n\}_{n=1}^{\infty}$ *for* H *a dual* K*-frame of* $\{f_n\}_{n=1}^{\infty}$ *if*

$$Kf = \sum_{n=1}^{\infty} \langle f, g_n \rangle f_n, \qquad \forall f \in H.$$

Definition 1.6. *A* K*-frame* $\{f_n\}_{n=1}^{\infty}$ *of* H *is called a* K*-exact frame if for every* $m \in I$ *(* I *is the countable index set) the sequence* $\{f_n\}_{n \neq m}$ *is not a* K*-frame for* H*. Also we call* $\{f_n\}_{n=1}^{\infty}$ *a* K*-minimal frame whenever for each* $\{c_n\} \in l^2$ *such that* $\sum_{n=1}^{\infty} c_n f_n = 0$ *then* $c_n = 0$ *for all* n*.*

2. Main results

We present condition on frames and K-frames to have duals. In bellow offer theorem and equivalent conditions for existance uniqeness dual for frames and present relationship between frame and Riesz basis.

Theorem 2.1. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a frame for* H*. Then the following are equivalent:*
(1) $\{f_n\}_{n=1}^{\infty}$ *is a Riesz basis for* H*.*
(2) If $\sum_{n=1}^{\infty} c_n f_n = 0$ *for some* $\{c_n\}_{n=1}^{\infty} \in l^2(\mathbb{N})$*, then* $c_n = 0$*,* $\forall n \in \mathbb{N}$*.*

Theorem 2.2. *If* $\{f_n\}_{n=1}^{\infty}$ *is a Riesz basis for* H*, then* $\{f_n\}_{n=1}^{\infty}$ *is a Bessel sequence. Furthermore, there exists a unique sequence* $\{g_n\}_{n=1}^{\infty}$ *in* H *such that*

$$f = \sum_{n=1}^{\infty} \langle f, g_n \rangle f_n, \qquad \forall f \in H.$$

The sequence $\{g_n\}_{n=1}^{\infty}$ *is also a Riesz basis, and the series converges unconditionally for all* $f \in H$*.*

Proposition 2.3. *Let* $\{f_n\}_{n=1}^{\infty}$ *be a frame for Hilbert space* H*. Then the following are equevalent:*
(1) $\{f_n\}_{n=1}^{\infty}$ *is a Riesz basis for* H*.*
(2) $\{f_n\}_{n=1}^{\infty}$ *is an exact frame, i.e. It ceases to be a frame when an arbitrary element is*

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removed.
(3) $\{f_n\}_{n=1}^{\infty}$ *is minimal, i.e.* $f_n \notin \overline{\mathrm{span}\{n_m : m \neq n\}}$ *for all n.*
(4) If $\sum_{n=1}^{\infty} c_n f_n = 0$ *for some* $\{c_n\} \in l^2$*, then* $c_n = 0$ *for all n.*
(5)$\{f_n\}_{n=1}^{\infty}$ *is a basis.*

Remark 2.4. *A frame that is not a Riesz basis is said to be overcomplete; in fact, if* $\{f_n\}_{n=1}^{\infty}$ *is a frame that is not a Riesz basis, there exist coefficient* $\{c_n\}_{n=1}^{\infty} \in l^2(\mathbb{N}) \setminus \{0\}$ *for which*

$$\sum_{n=1}^{\infty} c_n f_n = 0.$$

That is, for such frames there is some dependancy between the frame elements.

With some attention on which said above about frame and Riesz basis, we can present this corollary:

Corollary 2.5. *Let a frame* $\{f_n\}_{n=1}^{\infty}$ *be a Riesz basis, there is a unique Bessel sequence* $\{g_n\}_{n=1}^{\infty}$ *that be unique dual frame for* $\{f_n\}_{n=1}^{\infty}$*. Then* $\forall f \in H$

$$f = \sum_{n=1}^{\infty} \langle f, g_n \rangle f_n.$$

In K-frames the unique dual exists in the condition, that present bellow:

Lemma 2.6. *Every* K*-exact frame is a* K*-minimal frame, the convers does not hold in general.*

With the following example, we can show that the convers is not hold.

Example 2.7. *Let* $H = \mathbb{C}^3$ *and* $\{e_n\}_{n=1}^{\infty}$ *is the orthogonal basis of* H*. Define* $K : H \to H$ *by*

$$K \sum_{n=1}^{\infty} c_n e_n = c_1 e_1 + c_2 e_1 + c_3 e_2.$$

Then $K \in B(H)$ *and* $\{f_n\}_{n=1}^{\infty}$ *is a* K*-minimal frame. Easily, we can see that* $\{e_1, e_2\}$ *is also a* K*-frame with bounds* $A = \frac{1}{8}$ *and* $B = 1$*.*

The Riesz basis are in particular frames. Different from frames, each Riesz basis has a unique dual which is the canonical dual. Theorem 2.8 shows that the K-minimal frames also have such property.

Theorem 2.8. *A* K*-frame* $\{f_n\}_{n=1}^{\infty}$ *has a unique* K*-dual if and only if it is a* K*-minimal frame.*

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