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Continuous Wavelet Transform on a Quaternionic Hilbert Space

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Abstract

In this paper the new concept of the quaternionic unitary representation of a locally compact group to the unitary group of a quaternionic Hilbert space is studied. A continuous wavelet transform by means of a special case of such representations is defined to extend the continuous wavelet transform related to semidirect product groups.

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1. Introduction and Preliminaries

The theory of wavelet transforms is closely related to the theory of unitary representations of locally compact groups on the unitary group of a Hilbert space. In this paper, instead of a complex Hilbert space, we use a quaternionic Hilbert space and define a quaternionic unitary representation from a locally compact group to the unitary group of the quaternionic Hilbert space. Generalizing the results of [5], we prove some new facts regarding the irreducibility of a special kind of a quaternionic representation that is introduced in this paper. Quaternionic wavelet transform is highly applicable; according to [5], the results of [5] as well as the results of this paper, could be applicable in studying stereophonic or stereoscopic signals. Also, color images can be modeled by two-dimensional quaternion valued functions, so the quaternionic wavelet transform are useed in color image processing. We refer the interested reader to [1] and [2] for more information about quaternionic Fourier and wavelet transforms.
Throughout this paper, $\mathbb{H}$ will stand for the noncummutative field of quaternions and $\mathbb{H}^* = \mathbb{H} - \{0\}$. A quaternion q is of the form $q = x_0 + x_1 i + x_2 j + x_3 k$, where x_0, x_1, x_2 and $x_3 \in \mathbb{R}$ and i, j and k are the imaginary units and obey the following multiplication rules:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad \text{and} \quad ki = -ik = j.$$

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The quaternionic conjugate of q is $\overline{q} = x_0 - x_1 i - x_2 j - x_3 k$ and the norm of the quaternion q is $|q| = \sqrt{x_0^2 + x_1^2 + x_3^2 + x_4^2}$. It will be convenient to consider a quaternion q as a linear combination of two complex numbers $q = (x_0 + x_1 i) + (x_2 + x_3 i) j$. Therefore, $\bar{q} = \overline{(x_0 + x_1 i)} - (x_2 + x_3 i) j$ and $|q|^2 = |x_0 + i x_1|^2 + |x_2 + x_3 i|^2$.
Let H be a linear vector space over the field of quaternions under the right scalar multiplication. We suppose that a function $\langle . | . \rangle_{\mathsf{H}} : \mathsf{H} \times \mathsf{H} \longrightarrow \mathbb{H}$, which is called an inner product, exists such that for every $u, v, w \in \mathsf{H}$ and $p, q \in \mathbb{H}$ the following properties hold:
(i) $\overline{\langle u|v \rangle_{\mathsf{H}}} = \langle v|u \rangle_{\mathsf{H}}$;
(ii) $\langle u|u \rangle_{\mathsf{H}} > 0$ unless $u = 0$;
(iii) $\langle u|vp + wq \rangle_{\mathsf{H}} = \langle u|v \rangle_{\mathsf{H}} p + \langle u|w \rangle_{\mathsf{H}} q$.
If the space H is a complete metric space with the metric generated by the quaternionic norm $\|u\|_{\mathsf{H}} = \sqrt{\langle u|u \rangle_{\mathsf{H}}}$, then H is called a right quaternionic Hilbert space and it shares most of the properties of a complex Hilbert space, including Cauchy- Schwartz inequality and triangle inequality (see [4]).

Example 1.1. *For a measure space (X, μ_X), the space of all quaternion valued square integrable functions on X, which is shown by $L^2_{\mathbb{H}}(X, \mu_X)$, is a right quaternionic Hilbert space with the inner product*

$$\langle f|g \rangle_{L^2_{\mathbb{H}}(X,\mu_X)} = \int_X \langle f|g \rangle_{\mathbb{H}} d\mu_X = \int_X \overline{f(x)} g(x) d\mu_X(x), \qquad (1)$$

and the right scalar multiplication is defined by $(fq)(x) = f(x)q$, for $q \in \mathbb{H}$.

For a right quaternionic Hilbert space H, it is said that $T : \mathsf{H} \longrightarrow \mathsf{H}$ is a right linear operator if for all $u, v \in \mathsf{H}$ and $p \in \mathbb{H}$,

$$T(up + v) = (Tu)p + Tv.$$

Such an operator is said to be bounded if there exists $K \geq 0$ such that for all $u \in \mathsf{H}$,

$$\|Tu\|_{\mathsf{H}} \leq K \|u\|_{\mathsf{H}}.$$

The set of all bounded right linear operators on H is denoted by $\mathfrak{B}(\mathsf{H})$. The norm of a bounded right linear operator T is defined similar to the complex case and $\mathfrak{B}(\mathsf{H})$ is a complete normed space (see [4]; Proposition 2.11, for more properties of $\mathfrak{B}(\mathsf{H})$). In what follows, by an operator we mean a bounded right linear operator. Adjoint of an operator as well as unitary and positive operators are defined similar to the complex case. Many properties of the adjoint operator, are stated and proved in Theorem 2.15 and Remark 2.16 of [4], including $\|T\|_{\mathfrak{B}(\mathsf{H})} = \|T^*\|_{\mathfrak{B}(\mathsf{H})}$. It is emphasised that, the adjoint operation is not an involution on $\mathfrak{B}(\mathsf{H})$, as the equality $(qT)^* = \overline{q} T^*$ holds only for $q \in \mathbb{R}$. The set of unitary operators of $\mathfrak{B}(\mathsf{H})$ with composition, forms a group which is called the unitary group of H and is shown by $\mathcal{U}(\mathsf{H})$. See [4] for more properties of unitary and positive operators on H.

2. Quaternionic Unitary Representation

Let G be a locally compact group, H a right quaternionic Hilbert space and $\mathcal{U}(\mathsf{H})$ the unitary group of unitary operators on H. A quaternionic unitary representation of G is a homomorphism $\mathbf{U}_{\mathbb{H}}$ from G into $\mathcal{U}(\mathsf{H})$– that is a map $\mathbf{U}_{\mathbb{H}} : G \to \mathcal{U}(\mathsf{H})$ satisfying $\mathbf{U}_{\mathbb{H}}(xy) = \mathbf{U}_{\mathbb{H}}(x)\mathbf{U}_{\mathbb{H}}(y)$ and $\mathbf{U}_{\mathbb{H}}(x^{-1}) = \mathbf{U}_{\mathbb{H}}(x)^{-1} = \mathbf{U}_{\mathbb{H}}(x)^{*}$– such that $x \to \mathbf{U}_{\mathbb{H}}(x)u$ is continuous from G to H, for any $u \in \mathsf{H}$. A closed subspace $\mathcal{M}$ of H is called an invariant subspace for $\mathbf{U}_{\mathbb{H}}$ if $\mathbf{U}_{\mathbb{H}}(g)(\mathcal{M}) \subseteq \mathcal{M}$, for all $g \in G$. If $\mathbf{U}_{\mathbb{H}}$ admits a non trivial invariant subspace, then $\mathbf{U}_{\mathbb{H}}$ is called reducible, otherwise it is called irreducible.
Similar to the complex case, a quaternionic unitary representation $\mathbf{U}_{\mathbb{H}}$ of G on H is called square-integrable if there exists a non zero element $\varphi \in \mathsf{H}$ such that

$$\int_G |\langle \varphi | \mathbf{U}_{\mathbb{H}}(g)\varphi \rangle_{\mathsf{H}}|^2 \, d\mu_G(g) < \infty. \tag{2}$$

A unit vector φ satisfying (2) is said to be an admissible wavelet for $\mathbf{U}_{\mathbb{H}}$, and the constant

$$c_\varphi = \int_G |\langle \varphi | \mathbf{U}_{\mathbb{H}}(g)\varphi \rangle_{\mathsf{H}}|^2 \, d\mu_G(g), \tag{3}$$

is called the wavelet constant associated to the admissible wavelet φ.

In the following proposition by means of a unitary representation of G on the complex Hilbert space $L^2(X, \mu_X)$, a quaternionic unitary representation is built.

Proposition 2.1. *Let π be a unitary representation of the locally compact group G into $\mathcal{U}(L^2(X, \mu_X))$, for a measure space (X, μ_X). Define $\boldsymbol{U}_{\mathbb{H}} : G \to \mathcal{U}(L^2_{\mathbb{H}}(X, \mu_X))$, by*

$$\boldsymbol{U}_{\mathbb{H}}(g)f = \pi(g)f_1 + \pi(g)f_2 j, \quad f \in L^2_{\mathbb{H}}(X, \mu_X), \tag{4}$$

where $f = f_1 + f_2 j$, for two complex valued functions f_1 and f_2 in $L^2(X, \mu_X)$. Then
(i) $\boldsymbol{U}_{\mathbb{H}}$ is a quaternionic unitary representation;
(ii) If π is irreducible, then so is $\boldsymbol{U}_{\mathbb{H}}$;
(iii) π is square-integrable if and only if $\boldsymbol{U}_{\mathbb{H}}$ is.

3. Quaternionic Continuous Wavelet Transform

Let $G = S\sigma T$ be the semidirect product of two locally compact groups S and T, where S is abelian and $t \mapsto \sigma_t$ is a homomorphism from T into the group of automorphisms of S. Let $(s, t) \mapsto \sigma_t(s)$ be a continuous mapping from $S \times T$ onto S, then G is a locally compact group with the Cartesian product topology. By the transitive action of G on S defined by $(a, b).s = a\sigma_b(s)$, the group S is a homogeneous space of G. One can see [3] for related definitions.
Let $\tilde{T} = \{(e_1, t), t \in T\}$, where e_1 is the identity element of S. The rho-function for the pair $(G, \tilde{T})$, that will be shown by $\rho(t)$ during this paper, is given by $\rho(t) = \dfrac{\Delta_{\tilde{T}}(e_1, t)}{\Delta_G(e_1, t)}$, where $\Delta_{\tilde{T}}$ and Δ_G are modular functions of $\tilde{T}$ and G, respectively. If π from G into $\mathcal{U}(L^2(S, \mu_S))$, is the unitary representation

$$\left[\pi(a, b)f\right](s) = \rho(b)^{-1/2} f((a, b)^{-1}.s) = \rho(b)^{-1/2} f(\sigma_{b^{-1}}(a^{-1}s)), \tag{5}$$

the continuous wavelet transform (CWT) of $f \in L^2(S, \mu_S)$ is defined by

$$W_\psi f(s,t) = \langle \pi(s,t)\psi | f \rangle_{L^2(S,\mu_S)},$$

in which, $\psi \in L^1 \cap L^2(S, \mu_S)$ is a wavelet. This means that $\|\psi\|_{L^2(S,\mu_S)} = 1$ and there is a finite and non negative constant C_ψ such that for any $\gamma \in \hat{S}$, the dual group of S,

$$C_\psi = \int_T |\hat{\psi}(\gamma \circ \sigma_t)|^2 d\mu_T(t).$$

Definition 3.1. *Let π be the representation defined by (5), ψ the wavelet defined by (3), and $\Psi = \psi + \psi j$. The quaternionic CWT of $f = f_1 + f_2 j \in L^2_{\mathbb{H}}(S, \mu_S)$ at $(a,b) \in G$ is defined by*

$$W^{\mathbb{H}}_\Psi f(a,b) = \langle U_{\mathbb{H}}(a,b)\Psi | f \rangle_{L^2_{\mathbb{H}(S,\mu_S)}},$$

where, $\boldsymbol{U}_{\mathbb{H}}$ from $G = S\sigma T$ into the unitary group of $L^2_{\mathbb{H}}(S, \mu_S)$ is considered as

$$[\boldsymbol{U}_{\mathbb{H}}(a,b)f](s) = [\pi(a,b)f_1 + \pi(a,b)f_2 j](s).$$

In the following proposition we extend some of the well-known properties of the CWT to the quaternionic one.

Proposition 3.2. *By the notations of Definition 3.1,*
(i) $W^{\mathbb{H}}_\Psi f(a,b) = [(W_\psi f_1 + W_\psi f_2) + (W_\psi f_2 - \overline{W_\psi f_1})j](a,b)$;
(ii) $\langle W^{\mathbb{H}}_\Psi f | W^{\mathbb{H}}_\Psi g \rangle_{L^2_{\mathbb{H}}(G,\mu_G)} = 2C_\psi \langle f | g \rangle_{L^2_{\mathbb{H}}(S,\mu_S)}$;
(iii) $\|W^{\mathbb{H}}_\Psi f\|^2_{L^2_{\mathbb{H}}(G,\mu_G)} = 2C_\psi \|f\|^2_{L^2_{\mathbb{H}}(S,\mu_S)}$.

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