

On ternary amenable commutative JB* triple systems

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Abstract

In this paper we introduce the terms of triple approximate identity and ternary amenability for JB^* triple systems and then we show that every ternary amenable commutative JB^* triple system possesses an triple approximate identity.

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1. Introduction

JB^{*}-triples arose in the study of bounded symmetric domains in Banach spaces. A JB^{*}-triple system is a complex Banach space *M* together with a continuous triple product $\{\cdot, \cdot, \cdot\}: M \times M \times M \to M$ satisfying the following conditions:

(1) $\{\cdot, \cdot, \cdot\}$ is symmetric and bilinear in the outer two variables and conjugate-linear in middle variable;

 $(2)\{\cdot, \cdot, \cdot\}$ obeys the so-called Jordan identity

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y\{a, b, z\}\}, orequivalently,$$

 $\phi(a,b)\phi(x,y) - \phi(x,y)\phi(a,b) = \phi(\phi(a,b)x,y) - \phi(x,\phi(b,a)y),$

for all *a*, *b*, *x*, *y*, *z* \in *M*, where $\phi(a, b)(x) := \{a, b, x\};$

(3) for each $a \in M$, the operator $\phi(a, a)$ from *M* to *M* is a hermitian operator with non-negative spectrum;

(4) $||\{x, x, x\}| = ||x||^3$ for each $x \in M$.

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Note that $\phi(a, b)$ is sometimes denoted by $a \Box b$ and operators of this form are called box operators.

A JB^{*} triple system *M* is called commutative if the box operators $a \Box b$ and $x \Box y$ commute for all $a, b, x, y \in M$. It can be shown that *M* is commutative if and only if

$$\{a, b, \{x, y, z\}\} = \{a, \{b, x, y\}, z\} = \{\{a, b, x\}, y, z\}, a, b, x, y, z \in M$$

 C^* -algebras and JB*-algebras are the main examples of JB*-triples. More precisely, every C^* -algebra (resp. every JB*-algebra) is a JB*-triple with respect to the product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$ (resp. $\{x, y, z\} = (x \circ y^*) \circ z + (z \circ y^*) \circ x - (x \circ z) \circ y^*$). An example of a commutative JB* triple is any C^* - algebra with the triple product defined by $\{x, y, z\} = xy^*z$. The basic facts about JB*-triples can be found in [2] and some of the references therein.

Definition 1.1. Following the terms used in [3], a triple M-module is a vector space X equipped with three mappings

$$\{\cdots\}_l : X \times E \times E \to X, \ \{\cdots\}_m : E \times X \times E \to X$$

and

$$\{\cdots\}_r : E \times E \times X \to X$$

satisfying:

- {···}_l is linear in first and second variable and conjugate linear in last variable, {···}_m is conjugate trilinear and {···}_r is linear in middle and last variable and conjugate linear in first variable;
- **2** $\{x, b, a\}_l = \{a, b, x\}_r$ and $\{a, x, b\}_m = \{b, x, a\}_m$ for every $a, b \in M$ and $x \in X$;
- **3** $\{a, b, \{c, d, e\}\} = \{\{a, b, c\}, d, e\} \{d, \{b, a, d\}e\} + \{c, d, \{a, b, e\}\},\$ where $\{\cdots\}$ denotes any of the mapping $\{\cdots\}_l, \{\cdots\}_m, \{\cdots\}_r$ or the triple product of *M* and one of the elements *a*, *b*, *c*, *d*, *e* is in *X* and the rest are in *M*.

In [1], the triple M module defined above is called the triple M module of type I. It is called the triple M module of type II if the first item of the above definition is replaced by:

 $\{\cdots\}_l, \{\cdots\}_m$ and $\{\cdots\}_r$ are linear in outer variables and conjugate linear in middle variable.

It is also shown that the dual of a triple M-module of type I (resp II) is a triple M-module of type II (resp I). We usually write the expression triple M-module, without declaring the type, whenever a statement is true for both types or whenever the type is clear from the context.

Definition 1.2. Let *M* be a Jordan triple system and *X* be a triple *M*-module. A bounded linear operator $D: M \rightarrow X$ is said to be a **ternary derivation** if

 $D(\{abc\}) = \{D(a)bc\}_l + \{aD(b)c\}_m + \{abD(c)\}_r, \quad a, b, c \in M.$

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A ternary derivation $D : M \to X$ is said to be inner if there exists $a_i \in M$ and $x_i \in X$ for $i = 1, 2, \dots, n$ such that

$$D(b) = \sum_{i=1}^{n} (\{x_i a_i b\}_l - \{bx_i a_i\}_m)$$

Definition 1.3. A Jordan triple system M is called **ternary amenable**, if every bounded derivation $D: M \rightarrow X^*$ is inner for every Banach triple M-module X.

Definition 1.4. Let M be a Jordan triple system. A left-bounded approximate identity for M is a set of pair nets $\{(e_{\alpha}^{i}, u_{\alpha}^{i})_{\alpha \in I} : i = 1, 2, \dots, n\}$ in which for each i, $\{e^{i}\}_{\alpha}$ and $\{u^{i}\}_{\alpha}$ are bounded nets in M, that satisfies

$$\lim_{\alpha} (\sum_{i=1}^{n} (e_{\alpha}^{i} \Box u_{\alpha}^{i})) = id_{M}.$$

It is called a right-bounded approximate identity if

$$\lim_{\alpha} \sum_{i=1}^{n} (u_{\alpha}^{i} \Box e_{\alpha}^{i}) = id_{M}.$$

And it is called a middle-bounded approximate identity if

$$\lim_{\alpha} \sum_{i=1}^{n} Q(e_{\alpha}^{i}, u_{\alpha}^{i}) = id_{M}.$$

The set of $\{(e_{\alpha}^{i}, u_{\alpha}^{i})_{\alpha \in I} : i = 1, 2, \dots, n\}$ is called a bounded approximate identity for M if it is a left, right and middle-bounded approximate identity for M.

2. Main results

The following theorem establishes the relation between ordinary amenability of Banach algebras and ternary amenability:

Theorem 2.1. Let M be a Banach algebra and X a Banach M module. M is a Jordan triple system with the canonical triple product and X^* is a triple Banach M module. If M is ternary amenable then it is ordinary amenable.

A result of Dineen [4] shows that the bidual of a JB*-triple is again a JB*-triple. In [5], besides Dineen's approach, we gave an alternative construction, by mimicking Arens' method. This method leads us to the following theorem:

Theorem 2.2. Every ternary amenable commutative JB^* -triple system has a bounded triple approximate identity.

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