On ternary amenable commutative JB* triple systems

Amin Allah Khosravi*

Abstract

In this paper we introduce the terms of triple approximate identity and ternary amenability for JB* triple systems and then we show that every ternary amenable commutative JB* triple system possesses an triple approximate identity.

2010 *Mathematics subject classification:* Primary 46K70, Secondary 17C65..
Keywords and phrases: triple amenability, JB* triples, commutative JB* triples .

1. Introduction

JB*−triples arose in the study of bounded symmetric domains in Banach spaces. A JB*−triple system is a complex Banach space M together with a continuous triple product $\{\cdot,\cdot,\cdot\} : M \times M \times M \to M$ satisfying the following conditions:

(1) $\{\cdot,\cdot,\cdot\}$ is symmetric and bilinear in the outer two variables and conjugate-linear in middle variable;

(2) $\{\cdot,\cdot,\cdot\}$ obeys the so-called Jordan identity

$$\{a,b,\{x,y,z\}\} = \{\{a,b,x\},y,z\} - \{x,\{b,a,y\},z\} + \{x,y\{a,b,z\}\}, \text{ orequivalently,}$$

$$\phi(a,b)\phi(x,y) - \phi(x,y)\phi(a,b) = \phi(\phi(a,b)x,y) - \phi(x,\phi(b,a)y),$$

for all $a,b,x,y,z \in M$, where $\phi(a,b)(x) := \{a,b,x\}$;

(3) for each $a \in M$, the operator $\phi(a,a)$ from M to M is a hermitian operator with non-negative spectrum;

(4) $\|\{x,x,x\}| = \|x\|^3$ for each $x \in M$.

* speaker

Note that $\phi(a,b)$ is sometimes denoted by $a\Box b$ and operators of this form are called box operators.

A JB* triple system M is called commutative if the box operators $a\Box b$ and $x\Box y$ commute for all $a,b,x,y\in M$. It can be shown that M is commutative if and only if

$$\{a,b,\{x,y,z\}\} = \{a,\{b,x,y\},z\} = \{\{a,b,x\},y,z\}, \quad a,b,x,y,z\in M$$

C^*−algebras and JB*−algebras are the main examples of JB*−triples. More precisely, every C^*−algebra (resp. every JB*−algebra) is a JB*−triple with respect to the product $\{x,y,z\} = \frac{1}{2}(xy^*z+zy^*x)$ (resp. $\{x,y,z\} = (x\circ y^*)\circ z + (z\circ y^*)\circ x - (x\circ z)\circ y^*$). An example of a commutative JB* triple is any C^*− algebra with the triple product defined by $\{x,y,z\} = xy^*z$. The basic facts about JB*−triples can be found in [2] and some of the references therein.

Definition 1.1. *Following the terms used in [3], a triple M-module is a vector space X equipped with three mappings*

$$\{\cdots\}_l : X\times E\times E\to X, \quad \{\cdots\}_m : E\times X\times E\to X$$

and

$$\{\cdots\}_r : E\times E\times X\to X$$

satisfying:

1 *$\{\cdots\}_l$ is linear in first and second variable and conjugate linear in last variable, $\{\cdots\}_m$ is conjugate trilinear and $\{\cdots\}_r$ is linear in middle and last variable and conjugate linear in first variable;*

2 *$\{x,b,a\}_l = \{a,b,x\}_r$ and $\{a,x,b\}_m = \{b,x,a\}_m$ for every $a,b\in M$ and $x\in X$;*

3 *$\{a,b,\{c,d,e\}\} = \{\{a,b,c\},d,e\} - \{d,\{b,a,d\}e\} + \{c,d,\{a,b,e\}\}$, where $\{\cdots\}$ denotes any of the mapping $\{\cdots\}_l, \{\cdots\}_m, \{\cdots\}_r$ or the triple product of M and one of the elements a,b,c,d,e is in X and the rest are in M.*

In [1], the triple M module defined above is called the triple M module of type I. It is called the triple M module of type II if the first item of the above definition is replaced by:
$\{\cdots\}_l$, $\{\cdots\}_m$ and $\{\cdots\}_r$ are linear in outer variables and conjugate linear in middle variable.
It is also shown that the dual of a triple M-module of type I (resp II) is a triple M-module of type II (resp I). We usually write the expression triple M-module, without declaring the type, whenever a statement is true for both types or whenever the type is clear from the context.

Definition 1.2. *Let M be a Jordan triple system and X be a triple M-module. A bounded linear operator $D: M\to X$ is said to be a* ***ternary derivation*** *if*

$$D(\{abc\}) = \{D(a)bc\}_l + \{aD(b)c\}_m + \{abD(c)\}_r, \quad a,b,c\in M.$$

A ternary derivation $D : M \to X$ is said to be inner if there exists $a_i \in M$ and $x_i \in X$ for $i = 1, 2, \cdots, n$ such that

$$D(b) = \sum_{i=1}^{n} (\{x_i a_i b\}_l - \{b x_i a_i\}_m)$$

Definition 1.3. *A Jordan triple system M is called **ternary amenable**, if every bounded derivation $D : M \to X^*$ is inner for every Banach triple M-module X.*

Definition 1.4. *Let M be a Jordan triple system. A left-bounded approximate identity for M is a set of pair nets $\{(e^i_\alpha, u^i_\alpha)_{\alpha\in I} : i = 1, 2, \cdots, n\}$ in which for each i, $\{e^i\}_\alpha$ and $\{u^i\}_\alpha$ are bounded nets in M, that satisfies*

$$\lim_\alpha (\sum_{i=1}^{n} (e^i_\alpha \Box u^i_\alpha)) = id_M.$$

It is called a right-bounded approximate identity if

$$\lim_\alpha \sum_{i=1}^{n} (u^i_\alpha \Box e^i_\alpha) = id_M.$$

And it is called a middle-bounded approximate identity if

$$\lim_\alpha \sum_{i=1}^{n} Q(e^i_\alpha, u^i_\alpha) = id_M.$$

The set of $\{(e^i_\alpha, u^i_\alpha)_{\alpha\in I} : i = 1, 2, \cdots, n\}$ is called a bounded approximate identity for M if it is a left, right and middle-bounded approximate identity for M.

2. Main results

The following theorem establishes the relation between ordinary amenability of Banach algebras and ternary amenability:

Theorem 2.1. *Let M be a Banach algebra and X a Banach M module. M is a Jordan triple system with the canonical triple product and X^* is a triple Banach M module. If M is ternary amenable then it is ordinary amenable.*

A result of Dineen [4] shows that the bidual of a JB*-triple is again a JB*-triple. In [5], besides Dineen's approach, we gave an alternative construction, by mimicking Arens' method. This method leads us to the following theorem:

Theorem 2.2. *Every ternary amenable commutative JB*-triple system has a bounded triple approximate identity.*

References

[1] S. Dineen, The second dual of a JB*-triple system, *Complex Analysis, Functional Analysis and Approximation Theory* (Ed. J. Mujica), North-Holland, Amsterdam, (1986).

[2] T. Ho, A.M. Peralta and B. Russo, Ternary weakly amenable C*-algebras and JB*-triples, *Quart. J. Math.* **64** (2013), 1109-1139.

[3] A. A. Khosravi and H. R. Ebrahimi Vishki, Aron-Berner extensions of trilinear maps with application to the biduals of a JB*-triple, *preprint*

[4] M. Niazi, M.R. Miri and H.R. Ebrahimi Vishki, Ternary Weak Amenability of the Bidual of a JB*−Triple, to appear in *Banach J. Math. Anal.*

[5] B. Russo, Structure of JB*−triples, Jordan Algebras, *Proceedings of the Oberwolfach Conference 1992 (Ed. W. Kaup, K. Mc Crimmon and H. Petersson),* de Gruyter, Berlin, (1994), 209-280.

Amin Allah Khosravi,
Department of pure mathematics,
Ferdowsi university,
Mashhad, Iran
e-mail: amin.khosravi@stu-mail.um.ac.ir