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UEB-Topology on the Norm-Strict Bidual of Banach Algebras

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Abstract

For a Banach algebra A the strict topology β on A is defined as the locally convex topology on A induced by the family of seminorms $\{\lambda_a, a \in A\}$, where $\lambda_a(b) = \|ab\|$, for all $b \in A$. By the norm-strict bidual of A we mean the norm dual of the space $(A,\beta)^*$. In this paper, among other things, we investigate the continuity properties of the product on $((A,\beta)^*, \|\cdot\|)^*$, for some special Banach algebras. In particular, generalizing some results of M. Neufang, we show that if B is a norm bounded subset of $< A^*A >$, and if $m_0 \in RM(A)$, the right multiplier algebra of A, and $n_0 \in B$, then the mapping $(m,n) \to m\cdot n$ from $< A^*A > \times B$ into $< A^*A >$ is jointly continuous at (m_0,n_0) in the UEB- topology, that is the topology of uniform convergence on equicontinuous bounded subsets of $< A^*A >$.

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1. Introduction

For a Banach algebra A the (right) strict topology β on A is defined as the locally convex topology on A induced by the family of seminorms $\{\lambda_a, a \in A\}$, where $\lambda_a(b) = \|ab\|$, for all $b \in A$. The strict topology β on A is Hausdorff if and only if A is right faithful, i.e. $\{a \in A, ba = 0 \ forall\ b \in A\} = 0$. For example, any Banach algebra with a left approximate identity is right faithful. It is known that if A is right faithful, then the dual $(A,\beta)^*$ of A with respect to the strict topology β is the subspace $< A^*A >$, (i.e. the closed linear span of A^*A in the norm dual A^* of A). By Cohen's factorization theorem $< A^*A >= A^*A$ if A has a bounded right approximate identity. Similar results concerning the left strict topology are true.

Let A be a right faithful Banach algebra. By the norm-strict bidual of A we mean the norm dual of the space $(A,\beta)^* =< A^*A >$.

Let A^{**} denote the norm bidual of the Banach algebra A. A^{**} is a Banach algebra with both the first and second Arens products. The first Arens product on A^{**} is defined as follows:

$$< m\cdot n, f >=< m, n\cdot f >, \quad < n\cdot f, a >=< n, f\cdot a > \quad < f\cdot a, b >=< f, ab >$$

* speaker

for $m,n \in A^{**}$, $f \in A^*$ and $a,b \in A$. Let $q : A^{**} \to < A^*A >^*$ denote the canonical projection. Then $< A^*A >^*$ is a quotient algebra of A^{**} and, if A is right faithful, the mapping q is an injective algebra homomorphism which embeds A into $< A^*A >^*$. Since $< A^*A >^*$ is a quotient algebra of A^{**} with the first Arens product, therefore $< A^*A >^*$ turns to a Banach algebra with this product. The next result (needed in the sequel) is a generalization of Cohen's factorization theorem due to Sentilles and Taylor [2]. A similar result for the "right" case is correct.

Theorem 1.1. *(Sentilles-Taylor) Let A be a Banach algebra and let X be an essentisl (i.e. $AX = X$) left A-module. Assume further that A contains a bounded left approximate identity $\{e_\alpha\}$. Let $\mathcal{H} \subset X$ be a bounded set, then $\{e_\alpha\}$ is a uniform left approximate identity of $\mathcal{H}$, that is, $\lim_\alpha \sup_{x\in H} \|e_\alpha x - x\| = 0$ if and only if there exists $a \in A$ and $\mathcal{F} \subset X$ such that $\mathcal{H} = a\mathcal{F}$.*

From [2], a subset $\mathcal{H}$ of $< A^*A >$ is an equi-$< A^*A >$ set if there exists $a \in A$ such that $|f(x)| \le \|ax\|$ for all $f \in \mathcal{H}$ and $x \in A$. For a Bannach algebra A with a bounded approximate idenntity, the equi-$< A^*A >$ sets are explicitely characterized below. The following theorem is proved by Sentilles and Taylor [2] for general left Banach A-modules X.

Theorem 1.2. *(Sentilles-Taylor) Let A be a Banach algebra with a bai $(e_\alpha)_\alpha$. For a subset $\mathcal{H}$ of $(A,\beta)^* = A^*A$, the following statements are equivalent:*
(i) $\mathcal{H}$ is equi-A^*A,
(ii) $\mathcal{H}$ is norm bounded and $\lim_\alpha \sup_{f\in\mathcal{H}} \|f \cdot e_\alpha - f\| = 0$
(iii) There exist $a \in A$ and a subset $\mathcal{F} \subset Ball_1(A^*A)$ such that $\mathcal{H} = \mathcal{F} \cdot a$. ($Ball_1(A^*A)$ denotes the unit ball of A^*A.)

Most results of the present paper are generalizations of the results in [1].

2. The UEB-Topology

Let A be a Banach algebra with a bounded left approximate identity. Let $< A^*A >^*$ denote the norm dual of $< A^*A >$, or equivalently the norm-strict bidual of (A,β). Let UEB denote the topology of uniform convergence on equi-$< A^*A >$ subsets of $< A^*A >$, that is a net $(n_\alpha)_\alpha$ in $< A^*A >^*$ converges to 0 if and only if

$$\sup_{f\in\mathcal{H}} | < n_\alpha, f > | \to 0$$

for all equi-$< A^*A >$ subsets $\mathcal{H}$ of $< A^*A >$. Then we have the following theorem.

Theorem 2.1. *Let A be a Banach algebra with a bounded approximate identity. Then the UEB-topology on $(A^*A)^*$ coincides with the topology generated by the seminorms $\{\lambda_a, a \in A\}$ where $\lambda_a(n) = \|a \cdot n\|$ for all $n \in (A^*A)^*$.*

3. Continuity of the product in $(A^*A)^*$ with the UEB-topology

Let A be a right faithful Banach algebra. In this section, we introduce and study the UEB-topological center of the norm-strict bidual $< A^*A >^*$ of (A,β). First notice that if $n \in < A^*A >^*$ is an arbitrary element, then the right multiplication by n, $\rho_n :< A^*A >^* \to < A^*A >^*$ defined by $\rho_n(m) = m \cdot n$ is UEB-UEB continuous, i,e. if $m_\alpha \to 0$ in the UEB-topology of $< A^*A >^*$, then $m_\alpha \cdot n \to 0$ in the UEB-topology, since if $\mathcal{F}$ is equi-$< A^*A >$, so is the set $\{n \cdot f,\ f \in \mathcal{F}\}$. In fact, we have more as the next lemma shows.

Lemma 3.1. *Let A be a right faithful Banach algebra, $B \subset < A^*A >^*$ a norm bounded set and $\mathcal{F} \subset < A^*A >$ be an equi-$< A^*A >$ set. Then the set $\{n \cdot f,\ n \in B,\ f \in \mathcal{F}\}$ is equi-$< A^*A >$.*

Recall that the right strict topology $\beta = \beta_r$ on a Banach algebra A is defined as the locally convex topology on A induced by the family of seminorms $\{\lambda_a, a \in A\}$, where $\lambda_a(b) = \|ab\|$, for all $b \in A$. Also the left strict topology β_l on A is the locally convex topology on A induced by the family of seminorms $\{\rho_a, a \in A\}$, where $\rho_a(b) = \|ba\|$, for all $b \in A$.

Definition 3.2. *Let A be any Banach algebra. We say that A is a SIN Banach algebra if the right strict dual $(A,\beta = \beta_r)^*$ and the left strict dual $(A,\beta_l)^*$ of A coincide.*

Recall that A is called right (resp. left) faithful if $\{a \in A, ba = 0\ forall\ b \in A\} = 0$ (resp. $\{a \in A, ab = 0\ forall\ b \in A\} = 0$). A is faithful if it is both right and left faithful. For example, any Banach algebra with approximate identity is faithful.

Lemma 3.3. *Let A be a Banach algebra.*
(a) If A is right faithful, then $(A,\beta_r)^* =< A^*A >$.
(b) If A is left faithful, then $(A,\beta_l)^* =< AA^* >$.
(c) If A is faithful, then $(A,\beta_r)^* =< A^*A >$ and $< AA^* >= (A,\beta_l)^*$.
(d) A faithful Banach algebra A is SIN, provided $< A^*A >=< AA^* >$.
(e) A Banach algebra A with a bounded approximate identity is SIN, provided $A^*A = AA^*$.

It is evident from definition that a locally compact topological group G is a SIN group if and only its group algebra $L^1(G)$ is a SIN algebra.

Lemma 3.4. *Let A be a faithful Banach algebra and $\mathcal{F} \subset < A^*A >$ be an equi-$< A^*A >$ set. Then*
(i) The set $Ball_1(A) \cdot \mathcal{F} = \{a \cdot f, f \in \mathcal{F}, a \in A, \|a\| \le 1\}$ is equi-$< A^*A >$.
(ii) If A is a SIN Banach algebra, then the set $\mathcal{F} \cdot Ball_1(A) = \{f \cdot a, f \in \mathcal{F}, a \in A, \|a\| \le 1\}$ is equi-$< A^*A >$.

Definition 3.5. *Let A be a right faithful Banach algebra.The UEB-topological center $Z^{UEB}(< A^*A >^*)$ of the norm-strict bidual of (A,β) is defined by*

$$Z^{UEB}(< A^*A >^*) = \{m \in < A^*A >^*,\ \lambda_m :< A^*A >^* \to < A^*A >^* \ \ is\ UEB{-}UEB\ continuous\}.$$

It is easily seen that $A \subset Z^{UEB}(< A^*A >^*)$. In fact, for each $a \in A$, $\lambda_a(n)(f) =< n, f \cdot a >$, for all $n \in< A^*A >^*$ and $f \in< A^*A >^*$. So if $n_\alpha \to 0$ (in $< A^*A >^*$) with respect to the UEB-topology, then for each equi-$< A^*A >$ set $\mathcal{F}$,

$$\sup_{f\in\mathcal{F}} |\lambda_a(n_\alpha), f >| = \sup_{f\in\mathcal{F}} | < n_\alpha, f \cdot a > | \to 0$$

since the set $\mathcal{F} \cdot a$ is equi-$< A^*A >$. Therefore $A \subset Z^{UEB}(< A^*A >^*)$. For the case where $A = L^1(G)$ is the group algebra of a locally compact group G, it is proved that the UEB-topological center of $LUC(G)^*$, i.e. the norm-strict bidual of $(L^1(G), \beta)$, actually contains the measure algebra $M(G)$.

Proposition 3.6. *$RM(A) \subset Z^{UEB}(< A^*A >^*)$. In particular, if $A \cdot Z^t(< A^*A >^*) \subset A$ or equivalently $Z^t(< A^*A >^*) \subset RM(A)$ (by Neufang module maps prop. 2.3) , then $Z^t(< A^*A >^*) \subset Z^{UEB}(< A^*A >^*)$.*

The inclusion in the above proposition can be strict. In fact, if A is a SIN Banach algebra with $Z^t(< A^*A >^*) = RM(A)$ (like the group algebra of a locally compact SIN group G) (or A has a bounded right approximate identity which is not a right ideal in its second dual A^{**} but $A \cdot Z^t((A^*A)^*) \subset A$,) then

$$Z^{UEB}(< A^*A >^*) =< A^*A >^* \neq RM(A) = Z^t(< A^*A >^*).$$

Now the following theorem is easily verified. This is a generalization of a result of Neufang et. al. [1] when $A = L^1(G)$, for a locally compact group G.

Theorem 3.7. *Let A be a faithful Banach algebra, let $B \subset< A^*A >^*$ be a norm bounded set, $m_0 \in< A^*A >$ and $n_0 \in B$. Then the mapping $(m, n) \to m \cdot n$ from $< A^*A > \times B$ to $< A^*A >$ is jointly continuous at (m_0, n_0) in the UEB topology in each of these two cases:*
(a) $m_0 \in Z^{UEB}(< A^*A >^*)$ or in particular $m_0 \in RM(A)$,
(b)A is a SIN Banach algebra.

References

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