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Banach function algebras generated by normed vector spaces and linear functionals

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Abstract

Given a non-zero normed vector space A and a contractive non-zero element $\varphi \in A^*$ (the dual space of A), we introduce the Banach function algebra $C^\varphi(K)$, where $K = \overline{B_1^{(0)}}$ is the closed unit ball of A. We investigate and characterize some basic properties, such as idempotent elements, nilpotent elements and bounded approximate identities of the Banach function algebra $C^\varphi(K)$. Finally we characterize some elements of the character space $\triangle(C^\varphi(K))$.

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1. Introduction

Let A be a non-zero normed vector space and let φ be a non-zero contractive ($\|\varphi\| \leq 1$) element of A^*. Also let $K = \overline{B_1^{(0)}}$ be the closed unit ball of A. Suppose that $C(K) = \left\{f : K \longrightarrow \mathbb{C} \;\middle|\; f \text{ is continuous and bounded}\right\}$. Clearly $C(K)$ is a Banach space with respect to the norm, $\|f\|_\infty = \sup\left\{|f(x)| \;\middle|\; x \in K\right\}$.
For $f, g \in C(K)$ define,

$$(f \cdot g)(x) = f(x)g(x)\varphi(x), \quad x \in K.$$

We shall show that $(C(K), \cdot)$ is a non-unital commutative Banach algebra that we denote it by $C^\varphi(K)$. Also we characterize the idempotent, nilpotent elements and the bounded approximate identities of $C^\varphi(K)$. Finally we characterize some elements of the character space of $C^\varphi(K)$. Many basic properties of $C^\varphi(K)$ are investigated in [1].

2. Main results

In this section let A be a non-zero normed vector space and let φ be a non-zero linear functional on A with $\|\varphi\| \leq 1$. Also let $K = \overline{B_1^{(0)}} \subseteq A$.

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Proposition 2.1. *Let A be a non-zero normed vector space and let $\varphi \in A^*$ be a non-zero element such that $\|\varphi\| \leq 1$. Then $C^\varphi(K)$ is a non-unital commutative Banach algebra.*

Set $A^*\Big|_K = \Big\{f\Big|_K : K \longrightarrow \mathbb{C} \quad \Big| \quad f \in A^*\Big\}$. So we can present the following result.

Proposition 2.2. *Let A be a non-zero normed vector space and let $\varphi \in A^*$ be a non-zero element such that $\|\varphi\| \leq 1$. Then $A^*\Big|_K \subseteq C^\varphi(K)$ and for each $f \in A^*$, $\quad \|f\| = \|f\|_\infty$.*

In the following Proposition we characterize the form of idempotent elements of $C^\varphi(K)$.

Proposition 2.3. *Let A be a non-zero normed vector space and let $\varphi \in A^*$ be a non-zero element such that $\|\varphi\| \leq 1$. Then $f \in C^\varphi(K)$ is idempotent if and only if*

$$f(x) = \begin{cases} 0 & x \in f^{-1}(\{0\}) \\ \frac{1}{\varphi(x)} & x \notin f^{-1}(\{0\}) \end{cases}$$

Proposition 2.4. *Let A be a non-zero normed vector space and let $\varphi \in A^*$ be a non-zero element such that $\|\varphi\| \leq 1$. Then $f \in C^\varphi(K)$ is nilpotent if and only if $f\Big|_{K-\ker(\varphi)} = 0$.*

We give a result concerning bounded approximate identity.

Theorem 2.5. *Let A be a non-zero normed vector space and let $\varphi \in A^*$ be a non-zero element such that $\|\varphi\| \leq 1$. Then there is no bounded approximate identity in $C^\varphi(K)$.*

For each $k \in K$, define $\hat{k} : C^\varphi(K) \longrightarrow \mathbb{C}$ by $\hat{k}(f) = f(k), f \in C^\varphi(K)$. Obviously $K \subseteq C^\varphi(K)^*$.
Recall that an element $\psi \in C^\varphi(K)^*$ is a character if $\psi(f \cdot g) = \psi(f)\psi(g), f, g \in C^\varphi(K)$. Let $\triangle(C^\varphi(K))$ be the set of all characters on $C^\varphi(K)$. One can easily check that $\hat{0} \notin \triangle(C^\varphi(K))$. Indeed, As $1 \cdot 1 = \varphi\Big|_K$ then $\hat{0}(1 \cdot 1) = \hat{0}(\varphi\Big|_K) = \varphi(0) = 0 \neq \hat{0}(1)\hat{0}(1) = 1$.

Proposition 2.6. *Let A be a non-zero normed vector space and let $\varphi \in A^*$ be a non-zero element such that $\|\varphi\| \leq 1$. Then $K \cap \varphi^{-1}(\{1\}) \subseteq \triangle(C^\varphi(K))$.*

Remark 2.7. *Clearly $K \cap \ker(\varphi) \nsubseteq \triangle(C^\varphi(K))$. Indeed if $k \in K \cap \ker(\varphi)$ then $0 = \hat{k}(1 \cdot 1) \neq \hat{k}(1)\hat{k}(1) = 1$.*

In the sequel we present two questions.

Question 1. *Characterize the space $\triangle(C^\varphi(K))$.*

Question 2. *Characterize the Banach function algebra $C^\varphi(K)$ in the case where A is finite dimensional.*

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References

[1] A. R. Khoddami, On a certain class of Banach function algebras, preprint.

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