Submitted to the 5st Seminar on Harmonic Analysis and Applications
Organized by the Iranian Mathematical Society
January 18–19, 2017, Ferdowsi University of Mashhad, Iran

پنجمین سمینار
آنالیز هارمونیک
و کاربردها
5th
Seminar on
Harmonic Analysis
and Applications

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Certain Subspaces of the Dual of a Banach Algebra

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Abstract

The dual of an introverted subspace of the dual of a Banach algebra enjoys two (Arens type) products. In this talk we investigate the topological centers related to these products for a general introverted subspace. Some older results on certain introverted subspaces of $L^\infty(G)$ are extended to a general introverted subspace.

2010 *Mathematics subject classification:* Primary: 46H25.
Keywords and phrases: Arens product; topological center; introverted subspace.

1. Introduction

Following [1], the second dual A^{**} of a Banach algebra A enjoys two, in general, d multiplications (each turning A^{**} into a Banach algebra. In the case where these multiplications are coincide, A is said to be Arens regular.

Let A be a Banach algebra, A^* and A^{**} be the dual and the second dual of A, respectively. We shall make A^* into a Banach A-module under the module operations given by:

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle, \quad \langle a \cdot f, b\rangle = \langle f, ba\rangle.$$

A subspace X of A^* is called left (resp. right) invariant if $A\cdot X \subseteq X$ (resp. $X\cdot A \subseteq X$). A subspace X is called invariant, if it is both left and right invariant.

Let X be an invariant subspace of A^*, $m \in X^*$ and $f \in X$. We define, $m\Box f$ and $f\Diamond m$ so that, for every $a \in A$ $\langle m\Box f, a\rangle = \langle m, f\Box a\rangle$ and, $\langle a, f\Diamond m\rangle = \langle a\Diamond f, m\rangle$. Then X is called left (resp. right) introverted, if $X^*\Box X \subseteq X$ (resp. $X\Diamond X^* \subseteq X$); X is called introverted if it is both left and right introverted.

If X is left (resp. right) introverted, then $(X^*, \Box)$ (resp. $(X^*, \Diamond)$) is a Banach algebra under the product $m\Box n$ (resp. $m\Diamond n$), in which, $m\Box n$ and $m\Diamond n$ are defined so that, $\langle m\Box n, f\rangle = \langle m, n\Box f\rangle$, and $\langle f, m\Diamond n\rangle = \langle f\Diamond m, n\rangle$, for all $f \in X$. The products $\Diamond$ and $\Box$ are called the first and the second Arens (type) products on X^*, respectively. An introverted subspace X of A^* is called Arens (type) regular if $m\Box n = m\Diamond n$ for every $m, n \in X$.

* speaker

For a Banach algebra A it is obvious that A^* is introverted. In this case $\Box$ and $\Diamond$ are the so-called (first and second) Arens products on A^{**}, which makes A^{**} into a Banach algebras under each of these products. A less trivial example of a left (resp. right) introverted subspace of A^* is $A^*\Box A$ (resp. $A\Diamond A^*$), in the case where A enjoys a bounded approximate identity; [2].

If X is left introverted, then one may show that for every $n \in X^*$ the mapping $m \mapsto m\Box n$ is $w^* - w^*$ continuous, but $m \mapsto n\Box m$ is not continuous in general, unless n is in A. Whence the first topological center $Z_1(X^*)$ of X^* is defined so that,

$$Z_1(X^*) = \{n \in X^*; m \mapsto n\Box m \text{is} w^* - w^* - \text{continuous}\}.$$

If X right introverted, the second topological center $Z_2(X^*)$ of X^* is defined so that,

$$Z_2(X^*) = \{n \in A^{**}; m \mapsto m\Diamond n \text{is } w^* - w^* - \text{continuous}\}.$$

Trivially, $A \subseteq Z_1(X^*) \cap Z_2(X^*)$; and also $Z_1(X^*)$ and $Z_2(^*)$ are closed subalgebras of $(X^*, \Box)$ and $(X^*, \Diamond)$, respectively. We use the notations Z_1 and Z_2 for $Z_1(A^{**})$ and $Z_2(A^{**})$, respectively.

For the group algebra $A = L^1(G)$, in which G is a locally compact topological group, it is known that $Z_1 = Z_2 = L^1(G)$; [4]. Note that in this case $A^*\Box A = LUC(G)$ and $A\Diamond A^* = RUC(G)$. For $A = K(c_0)$, the operator algebra of all compact linear operators on the sequence space c_0, it has been shown that $Z_1 \neq Z_2$; [5].

2. The results

We start with the following elementary fact on the introversion ptoprty of invariant subspaces.

Proposition 2.1. *Every w^*-closed invariant subspace of A^* is introverted.*

A functional $f \in A^*$ is called weakly almost periodic if $f\Box A_1$ is weakly relatively compact in A^*. The set of all weakly compact elements of A^* will denote by $wap(A)$. It is easy to verify that $wap(A)$ is a closed subspace of A^*. The following result describes introversion of X in terms of the inclusion $X \subseteq wap(A)$.

Theorem 2.2. *For every norm closed translation invariant subspace X of A^*, the following are equivalent.*

(i) $X \subseteq wap(A)$.
(ii) X is left introverted and $Z_1(X^*) = X^*$.
(iii) X is right introverted and $Z_2(X^*) = X^*$.

As an immediate consequence we have the next corollary.

Corollary 2.3. *Every norm closed invariant subspace X of $wap(A)$ is introverted, and Arens (type) regular, i.e. $Z_1(X^*) = X^* = Z_2(X^*)$. In particular $wap(A)$ is introverted.*

The following result is an extension of [5, Corollay 3.2].

Theorem 2.4. *Let A be a Banach algebra and X be a left introverted subspace of A^* then*

(i) $X\Diamond Z_1(X^*) \subseteq X$.

(ii) If $X\Box A = X$, then $A\Diamond Z_1(X^*) \subseteq Z_1$.

(iii) if $X\Box A = A$ and $A.A = A$, then $A\Diamond Z_1(X^*) = A\Diamond Z_1$.

The next result extends [5, Theorem 3.6]) to a general left introverted subspace of A^*.

Theorem 2.5. *Let X be a norm closed left introverted subspace of A^* such that $X\Box A = X$, and $A.A = A$. Then the following statements are equivalents.*

(i)$X \subseteq wap(A)$.

(ii)$A\Diamond X^* \subseteq Z_1$.

(iii) $A\Diamond X^* \subseteq A\Diamond Z_1(X^*)$.

(iv) $Z_1(X^*) = X^*$.

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