

Certain Subspaces of the Dual of a Banach Algebra

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Abstract

The dual of an introverted subspace of the dual of a Banach algebra enjoys two (Arens type) products. In this talk we investigate the topological centers related to these products for a general introverted subspace. Some older results on certain introverted subspaces of $L^\infty(G)$ are extended to a general introverted subspace.

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1. Introduction

Following [1], the second dual A^{**} of a Banach algebra A enjoys two, in general, different multiplications (each turning A^{**} into a Banach algebra). In the case where these multiplications coincide, A is said to be Arens regular.

Let A be a Banach algebra, A^* and A^{**} be the dual and the second dual of A , respectively. We shall make A^* into a Banach A -module under the module operations given by:

$$\langle f \cdot a, b \rangle = \langle f, ab \rangle, \quad \langle a \cdot f, b \rangle = \langle f, ba \rangle.$$

A subspace X of A^* is called left (resp. right) invariant if $A \cdot X \subseteq X$ (resp. $X \cdot A \subseteq X$). A subspace X is called invariant, if it is both left and right invariant.

Let X be an invariant subspace of A^* , $m \in X^*$ and $f \in X$. We define, $m \square f$ and $f \diamond m$ so that, for every $a \in A$ $\langle m \square f, a \rangle = \langle m, f \square a \rangle$ and, $\langle a, f \diamond m \rangle = \langle a \diamond f, m \rangle$. Then X is called left (resp. right) introverted, if $X^* \square X \subseteq X$ (resp. $X \diamond X^* \subseteq X$); X is called introverted if it is both left and right introverted.

If X is left (resp. right) introverted, then (X^*, \square) (resp. (X^*, \diamond)) is a Banach algebra under the product $m \square n$ (resp. $m \diamond n$), in which, $m \square n$ and $m \diamond n$ are defined so that, $\langle m \square n, f \rangle = \langle m, n \square f \rangle$, and $\langle f, m \diamond n \rangle = \langle f \diamond m, n \rangle$, for all $f \in X$. The products \diamond and \square are called the first and the second Arens (type) products on X^* , respectively. An introverted subspace X of A^* is called Arens (type) regular if $m \square n = m \diamond n$ for every $m, n \in X$.

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For a Banach algebra A it is obvious that A^* is introverted. In this case \square and \diamond are the so-called (first and second) Arens products on A^{**} , which makes A^{**} into a Banach algebras under each of these products. A less trivial example of a left (resp. right) introverted subspace of A^* is $A^* \square A$ (resp. $A \diamond A^*$), in the case where A enjoys a bounded approximate identity; [2].

If X is left introverted, then one may show that for every $n \in X^*$ the mapping $m \mapsto m \square n$ is $w^* - w^*$ continuous, but $m \mapsto n \square m$ is not continuous in general, unless n is in A . Whence the first topological center $Z_1(X^*)$ of X^* is defined so that,

$$Z_1(X^*) = \{n \in X^*; m \mapsto n \square m \text{ is } w^* - w^* \text{ continuous}\}.$$

If X right introverted, the second topological center $Z_2(X^*)$ of X^* is defined so that,

$$Z_2(X^*) = \{n \in A^{**}; m \mapsto m \diamond n \text{ is } w^* - w^* \text{ continuous}\}.$$

Trivially, $A \subseteq Z_1(X^*) \cap Z_2(X^*)$; and also $Z_1(X^*)$ and $Z_2(X^*)$ are closed subalgebras of (X^*, \square) and (X^*, \diamond) , respectively. We use the notations Z_1 and Z_2 for $Z_1(A^{**})$ and $Z_2(A^{**})$, respectively.

For the group algebra $A = L^1(G)$, in which G is a locally compact topological group, it is known that $Z_1 = Z_2 = L^1(G)$; [4]. Note that in this case $A^* \square A = LUC(G)$ and $A \diamond A^* = RUC(G)$. For $A = K(c_0)$, the operator algebra of all compact linear operators on the sequence space c_0 , it has been shown that $Z_1 \neq Z_2$; [5].

2. The results

We start with the following elementary fact on the introversion property of invariant subspaces.

Proposition 2.1. *Every w^* -closed invariant subspace of A^* is introverted.*

A functional $f \in A^*$ is called weakly almost periodic if $f \square A_1$ is weakly relatively compact in A^* . The set of all weakly compact elements of A^* will denote by $wap(A)$. It is easy to verify that $wap(A)$ is a closed subspace of A^* . The following result describes introversion of X in terms of the inclusion $X \subseteq wap(A)$.

Theorem 2.2. *For every norm closed translation invariant subspace X of A^* , the following are equivalent.*

- (i) $X \subseteq wap(A)$.
- (ii) X is left introverted and $Z_1(X^*) = X^*$.
- (iii) X is right introverted and $Z_2(X^*) = X^*$.

As an immediate consequence we have the next corollary.

Corollary 2.3. *Every norm closed invariant subspace X of $wap(A)$ is introverted, and Arens (type) regular, i.e. $Z_1(X^*) = X^* = Z_2(X^*)$. In particular $wap(A)$ is introverted.*

The following result is an extension of [5, Corollary 3.2].

Theorem 2.4. *Let A be a Banach algebra and X be a left introverted subspace of A^* then*

- (i) $X \diamond Z_1(X^*) \subseteq X$.
- (ii) *If $X \square A = X$, then $A \diamond Z_1(X^*) \subseteq Z_1$.*
- (iii) *if $X \square A = A$ and $A.A = A$, then $A \diamond Z_1(X^*) = A \diamond Z_1$.*

The next result extends [5, Theorem 3.6]) to a general left introverted subspace of A^* .

Theorem 2.5. *Let X be a norm closed left introverted subspace of A^* such that $X \square A = X$, and $A.A = A$. Then the following statements are equivalent.*

- (i) $X \subseteq \text{wap}(A)$.
- (ii) $A \diamond X^* \subseteq Z_1$.
- (iii) $A \diamond X^* \subseteq A \diamond Z_1(X^*)$.
- (iv) $Z_1(X^*) = X^*$.

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