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Coherent Frames

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Abstract

In this paper we introduce a class of continuous frames in a Hilbert space $\mathcal{H}$ which is indexed by some locally compact group G, equipped with its left Haar measure. These frames are obtained as the orbits of a single element of Hilbert space $\mathcal{H}$ under some unitary representation π of G on $\mathcal{H}$.

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1. Introduction

In 1946 Gabor[4] introduced a method for reconstructing signals which led eventually to the theory of wavelets. Later in 1952 Duffin and Schaeffer introduced frame theory for Hilbert spaces to study some problems in nonharmonic Fourier series. Frames reintroduced in 1986 by Daubechies, Grossmann and Meyer[3]. Nowadays frames have become an alternative to orthonormal basis for reconstructing elements of a Hilbert space. Frames have been used in characterization of function spaces and other fields such as signal and image processing[2], filter bank theory and wireless communications.

The concept of generalization of frames to a family indexed by some measure space was proposed by G. Kaiser and independently by Ali, Antoine and Gazeau[1]. Kaiser used the terminology generalised frames. Also, in mathematical physics these frames are referred to as coherent states.

In this paper, we consider a continuous frame $\{f_g\}_{g\in G}$ indexed by a locally compact group G, equipped with the left Haar measure μ for which all the elements f_g appear by the action of G on a single element $f \in \mathcal{H}$ via a unitary represntation of G on $\mathcal{H}$ and study canonical dual and combinations of this frames.

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2. Basic Frame Theory

A countable family of elements $\{f_i\}_i$ in Hilbert space $\mathcal{H}$ is a discrete frame if there exist constants $A, B > 0$ such that for each $f \in \mathcal{H}$,

$$A\|f\|^2 \leq \sum_i \langle f, f_i \rangle \leq B\|f\|^2. \tag{1}$$

A and B are the lower and upper frame bounds. The frame $\{f_i\}_i$ is called a tight frame if $A = B$ and a normalized tight frame if $A = B = 1$. A uniform frame is a frame in which all the elements have equal norms. If the upper inequality in 1 holds, then $\{f_i\}_i$ is a Bessel sequence. If there is another frame $\{g_i\}_i \subset \mathcal{H}$ satisfying

$$f = \sum_i \langle f, f_i \rangle g_i, \quad \forall f \in \mathcal{H}$$

then $\{g_i\}_i$ is said to be a dual of $\{f_i\}_i$. Given a bessel sequence $\{f_i\}_i$, the synthesis operator $T : l^2 \to \mathcal{H}$ defined by

$$T\{c_i\}_i = \sum_i c_i f_i$$

is linear and bounded with $\|T\| \leq \sqrt{B}$. The adjoint of T is the analysis operator $T^* : \mathcal{H} \to l^2$ defined by

$$T^* f = \{\langle f, f_i \rangle\}_i.$$

The frame operator is $S = TT^*$, which is well defined and bounded. If $\{f_i\}_i$ is a frame, then $\{S^{-1} f_i\}_i$ is the canonical dual frame and every $f \in \mathcal{H}$ can be reconstructed as

$$f = \sum_i \langle f, f_i \rangle S^{-1} f_i.$$

A continuous frame for a Hilbert space $\mathcal{H}$ is a family $\{f_m\}_{m \in \mathcal{M}}$ indexed by a measure space $(\mathcal{M}, \mu)$ such that

- for all $f \in \mathcal{H}$, $m \to \langle f, f_m \rangle$ is a measurable function on $\mathcal{M}$;
- there exist constants $A, B > 0$ such that for each $f \in \mathcal{H}$

$$A\|f\|^2 \leq \int_M \langle f, f_m \rangle \mu(m) \leq B\|f\|^2.$$

The discrete frames correspond to the case where $\mathcal{M}$ is at most countable, equipped with the counting measure μ.

The continuous frame operator $S : \mathcal{H} \to \mathcal{H}$ is weakly defined by

$$\langle Sf, g \rangle = \int_{\mathcal{M}} \langle f, f_m \rangle \langle f_m, g \rangle \mu(m) \quad \forall f, g \in \mathcal{H}.$$

S is bounded, positive and invertible. Every $f \in \mathcal{H}$ has the representation

$$f = \int_{\mathcal{M}} \langle f, f_m \rangle S^{-1} f_m \mu(m)$$

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3. Coherent Frames

A unitary representation of a group G on a Hilbert space $\mathcal{H}$ is a linear mapping π of G on $\mathcal{H}$ such that $\pi(g)$ is a unitary operator for every $g \in G$.

Definition 3.1. *Let G be a locally compact abelian group. A coherent frame for a Hilbert space $\mathcal{H}$ is a continuous frame $\{\pi(g)\phi\}_{g\in G}$, where π is a unitary representation of G on $\mathcal{H}$ and $\phi \in \mathcal{H}$.*

Obviously, coherent frames are uniform. Before we develop the theory for coherent frames, we mention a few examples of coherent frames.

Example 3.2. *Let $G_{aff} = \mathbb{R} - \{0\} \times \mathbb{R}$ be affine group equipped with the measure $\frac{1}{a^2}ab$. If $\psi \in L^2(\mathbb{R})$ is admissible, i.e. , $C_\psi := \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\gamma)|^2}{|\gamma|}\gamma < \infty$, then the family $\{\psi^{a,b}\}_{(a,b)\in G_{aff}} = \{\pi(a,b)\psi\}_{(a,b)\in G_{aff}}$ is a cohernt tight frame with frame bound C_ψ for $L^2(\mathbb{R})$, where π is a unitary representation of G_{aff} on $L^2(\mathbb{R})$ defined by*

$$(\pi(a,b)f)(x) = (T_b D_a f)(x) = \frac{1}{\sqrt{|a|^2}} f(\frac{x-b}{a}),\ \ f \in L^2(\mathbb{R}), x \in \mathbb{R}.$$

Example 3.3. *Let $G = \mathbb{R}^2$ equipped with the lebesgue measure ab. If $g \in L^2(\mathbb{R}) - \{0\}$, then the family $\{g^{a,b}\}_{(a,b)\in G} = \{\pi(a,b)g\}_{(a,b)\in G}$ is a cohernt normalized tight frame, whrere π is a unitary representation of G on $L^2(\mathbb{R})$ defined by*

$$(\pi(a,b)g)(x) = (E_b T_a g)(x) = g(x-a)e^{2\pi ixb},\ \ x \in \mathbb{R}.$$

Now, we show that the canonical dual of a coherent frame is also a coherent frame.

Lemma 3.4. *Let $\{\pi(g)\phi\}_{g\in G}$ be a coherent frame and let S be its frame operator. Then S commutes with $\pi(g)$ for every $g \in G$.*

Proposition 3.5. *Let $\{\pi(g)\phi\}_{g\in G}$ be a coherent frame for $\mathcal{H}$. Then the canonical dual has the form $\{\pi(g)\psi\}_{g\in G}$ for some $\psi \in \mathcal{H}$.*

Recall that the frame operator S for a coherent frame is a positive invertible operator , and therefore has a positive square root operator $S^{\frac{1}{2}}$. The inverse of S is also a positive operator, and hence also has a squre root. The following proposition tells us that every coherent frame can be associated to a normalized tight coherent frame.

Proposition 3.6. *Let $\{\pi(g)\phi\}_{g\in G}$ be a coherent frame for $\mathcal{H}$. Then $\{S^{-\frac{1}{2}}\pi(g)\phi\}_{g\in G}$ is a coherent normalized tight frame for $\mathcal{H}$.*

In the rest of this paper, we show that coherent frames can be combined as follows:

- the direct sum of disjoint coherent frames is a coherent frame.
- the tensor product of coherent frames is a coherent frame.

Definition 3.7. *Let $\{\phi_m\}_{m\in\mathcal{M}}$ and $\{\psi_m\}_{m\in\mathcal{M}}$ be continuous frames for Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectivly. $\{\phi_m\}_{m\in\mathcal{M}}$ and $\{\psi_m\}_{m\in\mathcal{M}}$ are called disjoint if for all $x \in \mathcal{H}$ and for all $y \in \mathcal{K}$ we have $\int_{\mathcal{M}} \langle x, \phi_m\rangle_{\mathcal{H}} \langle \psi_m, y\rangle_{\mathcal{K}} \mu(m) = 0$.*

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Theorem 3.8. *Let* $\Phi = \{\pi(g)\phi\}_{g\in G}$ *and* $\Psi = \{\rho(g)\psi\}_{g\in G}$ *be coherent frames for Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$*, respectivly. The direct sum* $\Phi \oplus \Psi = \{\begin{pmatrix} \pi(g)\phi \\ \rho(g)\psi \end{pmatrix}\}_{g\in G}$ *is a coherent frame for direct sum Hilbert space* $\mathcal{H} \oplus \mathcal{K}$ *with frame operator* $S_{\Phi\oplus\Psi} = \begin{pmatrix} S_\Phi \\ S_\Psi \end{pmatrix}$ *if and only if* Φ *and* Ψ *are disjoint.*

Theorem 3.9. *Let* $\Phi = \{\pi(g)\phi\}_{g\in G}$ *and* $\Psi = \{\rho(g')\psi\}_{g'\in G'}$ *be coherent frames for Hilbert spaces* $\mathcal{H}$ *and* $\mathcal{K}$*, respectivly. The tensor product* $\Phi \otimes \Psi = \{\pi(g)\phi \otimes \rho(g')\psi\}_{g\in G, g'\in G'}$ *is a coherent frame for Hilbert tensor product space* $\mathcal{H} \otimes \mathcal{K}$*.*

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