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Wavelets method for solving a class of fractional partial integro-differential equations

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Abstract

In this paper, a Galerkin method based on the second kind Chebyshev wavelets (SKCWs) is proposed for solving a class of fractional partial integro-differential equations with weakly singular kernels. In the proposed method, the operational matrix of fractional order integration (OMFI) for SKCWs is used to transform the problem under consideration to a linear system of algebraic equations which can be simply solved.

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1. Introduction

The main goal of this paper is to propose an efficient and accurate method based on SKCWs to solve the following fractional partial integro-differential equation:

$$u_t(x,t) = f(x,t)+\mu\, u_{xx}(x,t)+\int_0^t (t-s)^{-\beta} u_{xx}(x,s)ds, \quad \mu \geq 0,\ 0<\beta<1,\ (x,t)\in\Omega, \quad (1)$$

with $\Omega = [0,1]\times[0,1]$, subject to the initial condition

$$u(x,0) = g(x), \quad (2)$$

and boundary conditions

$$u(0,t) = h_0(t), \qquad u(1,t) = h_1(t). \quad (3)$$

In the sequel, we use the following definition of the fractional calculus, which is required for establishing our results.

Definition 1.1. *The Riemann-Liouville fractional integration operator of order $\alpha \geq 0$ of a function $u(t)$ is defined in [1] by:*

$$\left(I^{\alpha}u\right)(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1}u(s)ds, & \alpha>0,\\ u(t), & \alpha=0. \end{cases} \quad (4)$$

* speaker

2. The HFs and SKCWs

An $\hat{m}$-set of the hat functions (HFs) is defined over the interval $[0,1]$ in [2]. An arbitrary function $f(t)$ defined on the interval $[0,1]$ may be expanded by the HFs as $f(t) \simeq \sum_{i=0}^{\hat{m}-1} f_i\varphi_i(t) = F^T\Phi(t) = \Phi(t)^T F$, where

$$F \triangleq [f_0, f_1, \ldots, f_{\hat{m}-1}]^T, \qquad \Phi(t) \triangleq [\varphi_0(t), \varphi_1(t), \ldots, \varphi_{\hat{m}-1}(t)]^T, \tag{5}$$

and $f_i = f\left(\frac{i}{\hat{m}-1}\right),\ i = 0, 1, \ldots, \hat{m}-1$.

Theorem 2.1. *(See [2]). The fractional integration of order α in the Riemann-Liouville sense of the vector $\Phi(t)$ can be expressed as:*

$$(I^\alpha\Phi)(t) \simeq \hat{P}^{(\alpha)}\Phi(t), \tag{6}$$

where matrix $\hat{P}^{(\alpha)}$ is called the OMFI of order α for the HFs.

The SKCWs are defined on the interval $[0,1]$ by [3]:

$$\psi_{nm}(t) = \begin{cases} \frac{2}{\sqrt{\pi}}2^{\frac{k}{2}}U_m\left(2^{k+1}t - 2n + 1\right), & t \in \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right], \\ 0, & o.w, \end{cases} \tag{7}$$

where $U_m(t)$ is the second kind Chebyshev polynomial of degree m. The set of the SKCWs is an orthogonal set on $[0,1]$ with respect to the weight function $w_n(t) = \begin{cases} \sqrt{1-(2^{k+1}t-2n+1)^2}, & t \in \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right], \\ 0, & o.w. \end{cases}$. A function $u(t)$ defined on $[0,1]$ may be expanded by the SKCWs as $u(t) \simeq \sum_{n=1}^{2^k}\sum_{m=0}^{M-1} c_{nm}\psi_{nm}(t) = C^T\Psi(t)$, where $c_{nm} = (u(t), \psi_{nm}(t))_{w_n(t)}$ and C and $\Psi(t)$ are $\hat{m} = 2^kM$ column vectors. It can be also written as $u(t) \simeq \sum_{i=1}^{\hat{m}} c_i\psi_i(t) = C^T\Psi(t)$, where $c_i = c_{nm}$ and $\psi_i(t) = \psi_{nm}(t)$, and the index i is determined by the relation $i = M(n-1) + m + 1$. Thus we have:

$$C \triangleq [c_1, c_2, \ldots, c_{\hat{m}}]^T, \qquad \Psi(t) \triangleq [\psi_1(t), \psi_2(t), \ldots, \psi_{\hat{m}}(t)]^T. \tag{8}$$

An arbitrary function of two variables $u(x,t)$, may be expanded by the SKCWs as $u(x,t) \simeq \sum_{i=1}^{\hat{m}}\sum_{j=1}^{\hat{m}} u_{ij}\psi_i(x)\psi_j(t) = \Psi^T(x)U\Psi(t)$, where $U = [u_{ij}]$ and $u_{ij} = \left(\psi_i(x), \left(u(x,t), \psi_j(t)\right)_{w_n(t)}\right)_{w_n(x)}$.

Theorem 2.2. *Let $\Phi(t)$ and $\Psi(t)$ be the HFs and SKCWs vectors defined in Eqs. (5) and (8), respectively. The vector $\Psi(t)$ can be expanded by the HFs vector $\Phi(t)$ as $\Psi(t) \simeq Q\Phi(t)$, where the $\hat{m} \times \hat{m}$ matrix Q is called the SKCWs matrix and*

$$Q_{ij} = \psi_i((j-1)h), \qquad i = 1, 2, \ldots, \hat{m},\ j = 1, 2, \ldots, \hat{m}. \tag{9}$$

Theorem 2.3. *The fractional integration of order $\alpha > 0$ in the Riemann-Liouville sense of the vector $\Psi(t)$ can be expressed as:*

$$(I^\alpha\Psi)(t) \simeq \left(Q\hat{P}^{(\alpha)}Q^{-1}\right)\Psi(t) \triangleq P^{(\alpha)}\Psi(t), \tag{10}$$

where matrix $P^{(\alpha)}$ is called the OMFI of order α for the SKCWs.

3. Description of the proposed method

To solve Eq. (1), we approximate $\frac{\partial^3 u(x,t)}{\partial x^2 \partial t}$ by SKCWs as follows:

$$\frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \simeq \Psi(x)^T U \Psi(t), \tag{11}$$

where $U = [u_{ij}]_{\hat{m}\times\hat{m}}$ should be found and $\Psi(.)$ is SKCWs vector defined in Eq. (8). By integrating of Eq. (11) two times with respect to x, we obtain:

$$u_t(x,t) \simeq u_t(0,t) + x\left(\frac{\partial u_t(x,t)}{\partial x}\mid_{x=0}\right) + \Psi(x)^T \left(P^T\right)^{(2)} U\Psi(t), \tag{12}$$

and by putting $x = 1$ in Eq. (12), and considering Eq. (2), we obtain:

$$\frac{\partial u_t(x,t)}{\partial x}\mid_{x=0} \simeq h_1'(t) - h_0'(t) - \Psi(1)^T \left(P^T\right)^{(2)} U\Psi(t). \tag{13}$$

We also expand $h_0'(t)$ and $h_1'(t)$ by SKCWs as:

$$h_0'(t) \simeq H_0^T \Psi(t), \qquad h_1'(t) \simeq H_1^T \Psi(t), \tag{14}$$

where H_0 and H_1 are coefficient vectors. Substituting Eq. (14) into Eq. (13) yields:

$$\frac{\partial u_t(x,t)}{\partial x}\mid_{x=0} \simeq \left(H_1^T - H_0^T - \Psi(1)^T \left(P^T\right)^{(2)} U\right)\Psi(t) \triangleq \widetilde{U}^T \Psi(t). \tag{15}$$

So, by substituting Eq. (15) into Eq. (12), we have:

$$u_t(x,t) \simeq \Psi(x)^T \left[E H_0^T + X\widetilde{U}^T + \left(P^T\right)^{(2)} U\right]\Psi(t) \triangleq \Psi(x)^T \Lambda_1 \Psi(t), \tag{16}$$

where X and E are coefficient vectors for x and the unit function, respectively. Now, by integrating of Eqs. (16) and (11) with respect to t, we obtain:

$$u(x,t) \simeq \Psi(x)^T \left[G_0 E^T + \Lambda_1 P\right]\Psi(t) \triangleq \Psi(x)^T \Lambda_2 \Psi(t), \tag{17}$$

$$u_{xx}(x,t) \simeq \Psi(x)^T \left[G_1 E^T + UP\right]\Psi(t) \triangleq \Psi(x)^T \Lambda_3 \Psi(t), \tag{18}$$

where G_0 and G_1 are coefficient vectors for $g(x)$ and $g''(x)$, respectively. Furthermore, we expand $f(x,t)$ as follows:

$$f(x,t) \simeq \Psi(x)^T F \Psi(t), \tag{19}$$

where F is coefficient matrices for $f(x,t)$. Then, by substituting Eqs. (16), (18) and (19) into Eq. (1), and using OMFI of SKCWs, we have:

$$\Psi(x)^T \left[\Lambda_1 - \mu\Lambda_3 - \Gamma(1-\beta)\Lambda_3 P^{(1-\beta)}\right]\Psi(t) \simeq \Psi(x)^T F \Psi(t). \tag{20}$$

It is obvious that Eq. (20) generates a set of $\hat{m}^2$ linear algebraic equations as:

$$\Lambda_1 - \mu\Lambda_3 - \Gamma(1-\beta)\Lambda_3 P^{(1-\beta)} = F. \tag{21}$$

Finally, by solving the above system for U, we obtain an approximate solution for the problem using Eq. (17).

Example 3.1. *Consider the fractional partial integro-differential equation [4]:*

$$u_t = \int_0^t (t-s)^{-\frac{1}{2}} u_{xx}(x,s)ds,$$

subject to the initial condition $u(x,0) = \sin(\pi x)$ *and the homogeneous boundary conditions. The exact solution of this problem is* $u(x,t) = \sum_{n=0}^{\infty}(-1)^n\Gamma\left(\frac{3}{2}n+1\right)^{-1}\left(\pi^{\frac{5}{2}}t^{\frac{3}{2}}\right)^n \sin(\pi x)$. *This problem is solved by the proposed method for* $k = 1$ *and* $M = 12$. *A comparison between the absolute errors obtained by the proposed method via the three-point explicit finite-difference method, the three-point implicit technique and the Crank-Nicolson procedure [4] with* $h = 0.02$ *at* $t = 1$ *is performed in Table 1.*

TABLE 1. The absolute errors obtained by the finite difference schemes and our method.

	Finite difference schemes			Our method
x	Explicit	Implicit	Crank-Nicolson	$k = 1, M = 12$
0.1	7.5×10^{-3}	7.1×10^{-3}	5.1×10^{-3}	1.4×10^{-3}
0.3	7.6×10^{-3}	7.4×10^{-3}	5.3×10^{-3}	3.7×10^{-3}
0.5	7.5×10^{-3}	7.5×10^{-3}	5.4×10^{-3}	4.5×10^{-3}
0.7	7.3×10^{-3}	7.2×10^{-3}	5.5×10^{-3}	3.7×10^{-3}
0.9	7.8×10^{-3}	7.6×10^{-3}	5.2×10^{-3}	1.4×10^{-3}

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