

Wavelets method for solving a class of fractional partial integro-differential equations

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Abstract

In this paper, a Galerkin method based on the second kind Chebyshev wavelets (SKCWs) is proposed for solving a class of fractional partial integro-differential equations with weakly singular kernels. In the proposed method, the operational matrix of fractional order integration (OMFI) for SKCWs is used to transform the problem under consideration to a linear system of algebraic equations which can be simply solved.

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1. Introduction

The main goal of this paper is to propose an efficient and accurate method based on SKCWs to solve the following fractional partial integro-differential equation:

$$u_t(x,t) = f(x,t) + \mu u_{xx}(x,t) + \int_0^t (t-s)^{-\beta} u_{xx}(x,s) ds, \ \mu \ge 0, \ 0 < \beta < 1, \ (x,t) \in \Omega, \ (1)$$

with $\Omega = [0, 1] \times [0, 1]$, subject to the initial condition

$$u(x,0) = g(x), \tag{2}$$

and boundary conditions

$$u(0,t) = h_0(t), \qquad u(1,t) = h_1(t).$$
 (3)

In the sequel, we use the following definition of the fractional calculus, which is required for establishing our results.

Definition 1.1. The Riemann-Liouville fractional integration operator of order $\alpha \ge 0$ of a function u(t) is defined in [1] by:

$$(I^{\alpha}u)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds, & \alpha > 0, \\ u(t), & \alpha = 0. \end{cases}$$
(4)

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2. The HFs and SKCWs

An \hat{m} -set of the hat functions (HFs) is defined over the interval [0, 1] in [2]. An arbitrary function f(t) defined on the interval [0, 1] may be expanded by the HFs as $f(t) \simeq \sum_{i=0}^{\hat{m}-1} f_i \varphi_i(t) = F^T \Phi(t) = \Phi(t)^T F$, where

$$F \triangleq [f_0, f_1, \dots, f_{\hat{m}-1}]^T, \qquad \Phi(t) \triangleq [\varphi_0(t), \varphi_1(t), \dots, \varphi_{\hat{m}-1}(t)]^T, \qquad (5)$$

and $f_i = f\left(\frac{i}{\hat{m}-1}\right), \ i = 0, 1, \dots, \hat{m} - 1.$

Theorem 2.1. (See [2]). The fractional integration of order α in the Riemann-Liouville sense of the vector $\Phi(t)$ can be expressed as:

$$(I^{\alpha}\Phi)(t) \simeq \hat{P}^{(\alpha)}\Phi(t), \tag{6}$$

where matrix $\hat{P}^{(\alpha)}$ is called the OMFI of order α for the HFs.

The SKCWs are defined on the interval [0, 1] by [3]:

$$\psi_{nm}(t) = \begin{cases} \frac{2}{\sqrt{\pi}} 2^{\frac{k}{2}} U_m \left(2^{k+1} t - 2n + 1 \right), & t \in \left[\frac{n-1}{2^k}, \frac{n}{2^k} \right], \\ 0, & o.w, \end{cases}$$
(7)

where $U_m(t)$ is the second kind Chebyshev polynomial of degree *m*. The set of the SKCWs is an orthogonal set on [0, 1] with respect to the weight function $w_n(t) =$

 $\begin{cases} \sqrt{1 - (2^{k+1}t - 2n + 1)^2}, & t \in \left[\frac{n-1}{2^k}, \frac{n}{2^k}\right], \\ 0, & o.w. \end{cases}$ A function u(t) defined on [0, 1] may

be expanded by the SKCWs as $u(t) \simeq \sum_{n=1}^{2^k} \sum_{m=0}^{M-1} c_{nm} \psi_{nm}(t) = C^T \Psi(t)$, where $c_{nm} = (u(t), \psi_{nm}(t))_{w_n(t)}$ and C and $\Psi(t)$ are $\hat{m} = 2^k M$ column vectors. It can be also written as $u(t) \simeq \sum_{i=1}^{\hat{m}} c_i \psi_i(t) = C^T \Psi(t)$, where $c_i = c_{nm}$ and $\psi_i(t) = \psi_{nm}(t)$, and the index *i* is determined by the relation i = M(n-1) + m + 1. Thus we have:

$$C \triangleq [c_1, c_2, \dots, c_{\hat{m}}]^T, \qquad \Psi(t) \triangleq [\psi_1(t), \psi_2(t), \dots, \psi_{\hat{m}}(t)]^T.$$
(8)

An arbitrary function of two variables u(x,t), may be expanded by the SKCWs as $u(x,t) \simeq \sum_{i=1}^{\hat{m}} \sum_{j=1}^{\hat{m}} u_{ij} \psi_i(x) \psi_j(t) = \Psi^T(x) U \Psi(t)$, where $U = [u_{ij}]$ and $u_{ij} = \left(\psi_i(x), \left(u(x,t), \psi_j(t)\right)_{w_n(x)}\right)_{w_n(x)}$.

Theorem 2.2. Let $\Phi(t)$ and $\Psi(t)$ be the HFs and SKCWs vectors defined in Eqs. (5) and (8), respectively. The vector $\Psi(t)$ can be expanded by the HFs vector $\Phi(t)$ as $\Psi(t) \simeq Q\Phi(t)$, where the $\hat{m} \times \hat{m}$ matrix Q is called the SKCWs matrix and

$$Q_{ij} = \psi_i((j-1)h), \quad i = 1, 2, \dots, \hat{m}, \ j = 1, 2, \dots, \hat{m}.$$
 (9)

Theorem 2.3. The fractional integration of order $\alpha > 0$ in the Riemann-Liouville sense of the vector $\Psi(t)$ can be expressed as:

$$(I^{\alpha}\Psi)(t) \simeq \left(Q\hat{P}^{(\alpha)}Q^{-1}\right)\Psi(t) \triangleq P^{(\alpha)}\Psi(t), \tag{10}$$

where matrix $P^{(\alpha)}$ is called the OMFI of order α for the SKCWs.

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3. Description of the proposed method

To solve Eq. (1), we approximate $\frac{\partial^3 u(x,t)}{\partial x^2 \partial t}$ by SKCWs as follows:

$$\frac{\partial^3 u(x,t)}{\partial x^2 \partial t} \simeq \Psi(x)^T U \Psi(t), \tag{11}$$

where $U = [u_{ij}]_{\hat{m} \times \hat{m}}$ should be found and $\Psi(.)$ is SKCWs vector defined in Eq. (8). By integrating of Eq. (11) two times with respect to *x*, we obtain:

$$u_t(x,t) \simeq u_t(0,t) + x \left(\frac{\partial u_t(x,t)}{\partial x} \Big|_{x=0} \right) + \Psi(x)^T \left(P^T \right)^{(2)} U \Psi(t),$$
(12)

and by putting x = 1 in Eq. (12), and considering Eq. (2), we obtain:

$$\frac{\partial u_t(x,t)}{\partial x}|_{x=0} \simeq h_1'(t) - h_0'(t) - \Psi(1)^T \left(P^T\right)^{(2)} U\Psi(t).$$
(13)

We also expand $h'_0(t)$ and $h'_1(t)$ by SKCWs as:

$$h'_{0}(t) \simeq H_{0}^{T} \Psi(t), \qquad h'_{1}(t) \simeq H_{1}^{T} \Psi(t),$$
(14)

where H_0 and H_1 are coefficient vectors. Substituting Eq. (14) into Eq. (13) yields:

$$\frac{\partial u_t(x,t)}{\partial x} \mid_{x=0} \simeq \left(H_1^T - H_0^T - \Psi(1)^T \left(P^T \right)^{(2)} U \right) \Psi(t) \triangleq \widetilde{U}^T \Psi(t).$$
(15)

So, by substituting Eq. (15) into Eq. (12), we have:

$$u_t(x,t) \simeq \Psi(x)^T \left[EH_0^T + X\widetilde{U}^T + \left(P^T \right)^{(2)} U \right] \Psi(t) \triangleq \Psi(x)^T \Lambda_1 \Psi(t),$$
(16)

where X and E are coefficient vectors for x and the unit function, respectively. Now, by integrating of Eqs. (16) and (11) with respect to t, we obtain:

$$u(x,t) \simeq \Psi(x)^T \left[G_0 E^T + \Lambda_1 P \right] \Psi(t) \triangleq \Psi(x)^T \Lambda_2 \Psi(t), \tag{17}$$

$$u_{xx}(x,t) \simeq \Psi(x)^T \left[G_1 E^T + UP \right] \Psi(t) \triangleq \Psi(x)^T \Lambda_3 \Psi(t), \tag{18}$$

where G_0 and G_1 are coefficient vectors for g(x) and g''(x), respectively. Furthermore, we expand f(x, t) as follows:

$$f(x,t) \simeq \Psi(x)^T F \Psi(t), \tag{19}$$

where F is coefficient matrices for f(x, t). Then, by substituting Eqs. (16), (18) and (19) into Eq. (1), and using OMFI of SKCWs, we have:

$$\Psi(x)^{T} \left[\Lambda_{1} - \mu \Lambda_{3} - \Gamma(1 - \beta) \Lambda_{3} P^{(1 - \beta)} \right] \Psi(t) \simeq \Psi(x)^{T} F \Psi(t).$$
⁽²⁰⁾

It is obvious that Eq. (20) generates a set of \hat{m}^2 linear algebraic equations as:

$$\Lambda_1 - \mu \Lambda_3 - \Gamma(1 - \beta) \Lambda_3 P^{(1 - \beta)} = F.$$
⁽²¹⁾

Finally, by solving the above system for U, we obtain an approximate solution for the problem using Eq. (17).

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Example 3.1. Consider the fractional partial integro-differential equation [4]:

$$u_t = \int_0^t (t-s)^{-\frac{1}{2}} u_{xx}(x,s) ds,$$

subject to the initial condition $u(x, 0) = \sin(\pi x)$ and the homogeneous boundary conditions. The exact solution of this problem is $u(x, t) = \sum_{n=0}^{\infty} (-1)^n \Gamma(\frac{3}{2}n+1)^{-1} (\pi^{\frac{5}{2}}t^{\frac{3}{2}})^n \sin(\pi x)$. This problem is solved by the proposed method for k = 1 and M = 12. A comparison between the absolute errors obtained by the proposed method via the three-point explicit finite-difference method, the three-point implicit technique and the Crank-Nicolson procedure [4] with h = 0.02 at t = 1 is performed in Table 1.

TABLE 1. The absolute errors obtained by the linite difference schemes and our method.				
	Finite d	Our method		
x	Explicit	Implicit	Crank-Nicolson	k = 1, M = 12
0.1	7.5×10^{-3}	7.1×10^{-3}	5.1×10^{-3}	1.4×10^{-3}
0.3	7.6×10^{-3}	7.4×10^{-3}	5.3×10^{-3}	3.7×10^{-3}
0.5	7.5×10^{-3}	7.5×10^{-3}	5.4×10^{-3}	4.5×10^{-3}
0.7	7.3×10^{-3}	7.2×10^{-3}	5.5×10^{-3}	3.7×10^{-3}
0.9	7.8×10^{-3}	7.6×10^{-3}	5.2×10^{-3}	1.4×10^{-3}

TABLE 1. The absolute errors obtained by the finite difference schemes and our method

References

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