Submitted to the 5st Seminar on Harmonic Analysis and Applications
Organized by the Iranian Mathematical Society
January 18–19, 2017, Ferdowsi University of Mashhad, Iran

پنجمین سمینار آنالیز هارمونیک و کاربردها 5th Seminar on Harmonic Analysis and Applications

# Creation and annihilation operators on de Sitter manifold

A. Rabeie*

**Abstract**

In this paper by using the Kirillov orbit method, we show that the phase space for a scalar massive particle on 1+1-de Sitter space is a cotangent bundle $T^*(S^1)$ which is isomorphic to the complex circle $S^1_{\mathbb{C}}$. The eigenstates in associated Hilbert space which are the coherent states, are obtained from the heat kernel of complex circle. By these states and the Berezin integral quantization, we obtain the creation and annihilation operators on de Sitter.

2010 *Mathematics subject classification:* Primary 34B45, Secondary 83D05.

*Keywords and phrases:* :Coherent States, orbit, quantization.

## 1. Introduction

Study of the simple harmonic oscillator in quantum mechanics is one of the most important application of the harmonic analysis in physics. The phase space of this motion is isomorphic to $\mathbb{C}$. For each point of this space we associate a vector in Hilbert space which have the three properties of Coherent states(CS). By using of this vector and the Berezin integral quantization, we can obtain the creation and annihilation operators for simple harmonic oscillator.

Another application of harmonic analysis is in general relativity. The general relativity is known by Einstein's equations in Minkowskian space (space-time):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu} - \Lambda\, g_{\mu\nu} = -8\pi G T_{\mu\nu}. \tag{1}$$

* speaker

In this equation, gravitational force is considered as curvature of space-time. The de Sitter metric ($g_{\mu\nu}$) is one of the solutions of this equation with null energy-momentum ($T_{\mu\nu} = 0$) and positive cosmological constant ($\Lambda > 0$). This metric is visualized by a hyperboloid embedded in a five-dimensional Minkowski space which is known as 1+3-de Sitter space-time. The quantum calculation for a movement particle on this space is very difficult and therefore we limit our discussion on 1+1-de Sitter space. In this work, we obtain the creation and annihilation operators for simple harmonics oscillator and a movement massive particle on 1+1-de Sitter space by Berezin quantization method [3]: $O_f = \int_X f(\chi)|\chi >< \chi|\,\mu(d\chi)$ ,
where $\mu(d\chi)$ and $|\chi >$ are respectively, the relevant measure on phase space($X$) and CS. Also, $f(\chi)$ is a classical observable (a function on phase space) .

## 2. Creation and annihilation operators for simple harmonics oscillator

The phase space of one particle which doing simple harmonics oscillation is isomorphic to the complex space $\mathbb{C}$. Correspond to each point of this space ($z = \frac{1}{\sqrt{2}}(q + ip) \in \mathbb{C}$), we can introduce a vector in Hilbert space. This vector is called the standard coherent state and given by [4] :

$$|z >= |q, p >= e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n > . \tag{2}$$

These states satisfy the resolution of the unity :

$$\int_{\mathbb{C}} \frac{d^2 z}{\pi} |z >< z| = \sum_{n=0}^{\infty} |n >< n| = I_d, \tag{3}$$

where $\frac{d^2 z}{\pi}$ is lebesgue measure and $\{|n >\}$ are orthonormal basis in Fock representation. Usually, having the resolution of the unity is equivalence to doing the Berezin's quantization on harmonics oscillator. Therefore, from eq. (1) we can obtain the creation and annihilation operators as follows:

$$O_{\bar{z}} = \int_{\mathbb{C}} \frac{d^2 z}{\pi}\, \bar{z}\, |z >< z| = \sum_{n=0}^{\infty} \sqrt{n+1} |n+1 >< n|, \tag{4}$$

$$O_z = \int_{\mathbb{C}} \frac{d^2 z}{\pi}\, z\, |z >< z| = \sum_{n=0}^{\infty} \sqrt{n} |n-1 >< n| . \tag{5}$$

### 3. Creation and annihilation operators on 1+1-de Sitter

The co-adjoint orbit is mathematically a phase space for a Hamiltonian system whose group G is the symmetric group. Moreover, for all simple or semi-simple Lie groups, one can identify the Lie algebra and its dual. Therefore, for de-Sitter group which is a simple group, the adjoint orbits of movement massive particle on 1+1-de Sitter is the phase space.

On the other hand, the symmetric covering group for 1+1-de Sitter space is group $SU(1,1)$ that is given by [1] :

$$SU(1,1) \ni g = \underbrace{\begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}}_{\text{“space translation ''}} \underbrace{\begin{pmatrix} \cosh\frac{\psi}{2} & \sinh\frac{\psi}{2} \\ \sinh\frac{\psi}{2} & \cosh\frac{\psi}{2} \end{pmatrix}}_{\text{“time translation ''}} \underbrace{\begin{pmatrix} \cosh\frac{\varphi}{2} & i\sinh\frac{\varphi}{2} \\ -i\sinh\frac{\varphi}{2} & \cosh\frac{\varphi}{2} \end{pmatrix}}_{\text{“Lorentz transformation ''}} . \qquad (6)$$

Group $SU(1,1)$ is simple and one can show that the associated phase space is identified with cotangent space $T^*(S^1)$. This cotangent bundle is isomorphic to the complex circle “$S^1_{\mathbb{C}}$ "(see [2] for $m = r = \hbar = 1$):

$$S^1_{\mathbb{C}} = \left\{ \overrightarrow{a} = \cosh(p)\overrightarrow{x} + i\,\frac{\sinh(p)}{p}\overrightarrow{p} \in \mathbb{C}^2 \right\} = \{a_1 = \cos(\beta + ip)\,,\; a_2 = \sin(\beta + ip)\}\;, \qquad (7)$$

where the $(\beta, p)$ play the role of pair varieties of $T^*(S^1)$ and $a^2 = x^2 = 1,\; p^2 = p_1^2 + p_2^2$.

In other words, we must construct our CS on $S^1_{\mathbb{C}}$. For this purpose, we use the heat kernel on complex circle ( i.e. $\rho^1_\tau(\overrightarrow{a}, \overrightarrow{x})$ ) which was presented by Hall-Mitchell [2]. This kernel is related to CS ($|\Psi^\tau_a>$) as follows :

$$\rho^1_\tau(\overrightarrow{a}, \overrightarrow{x}) =< x|\Psi^\tau_a >= \frac{1}{\sqrt{2\pi\tau}} \sum_{n=-\infty}^{\infty} e^{\frac{-(\tilde{\theta}-2\pi n)^2}{2\tau}}, \qquad \vec{a} \in S^1_{\mathbb{C}}\,,\; \vec{x} \in S^1\,,$$

where $\tau$ is a positive real parameter and $\tilde{\theta}$ is a complex angle with $0 \le Re\,\tilde{\theta} \le \pi$. By the Poisson summation “ $\sum_{n=-\infty}^{+\infty} f(n) = \sum_{n=-\infty}^{+\infty} \widehat{f}(n)$ " and “ $\cos\tilde{\theta} = \overrightarrow{x}.\overrightarrow{a} = \cos(ip)$ " we obtain :

$$\begin{aligned} |\overrightarrow{x}> &= \sum_n e^{-in\beta}|n>, \\ |\Psi^\tau_a> &= \frac{1}{\sqrt{\mathcal{N}}} \sum_n e^{-\frac{\tau n^2}{2}} e^{np} e^{-in\beta}|n>\;, \end{aligned} \qquad (8)$$

where $\mathcal{N} = \vartheta_3(\frac{p}{i}|\frac{i\tau}{\pi}) = \sum_n e^{-\tau n^2} e^{2np} < \infty$, $\quad \sum_n |n >< n| = I_d$ .
Also, by using the equation (1) and the measure " $\mu(d\beta, dp) = \frac{\mathcal{N}}{2\pi\sqrt{\pi\tau}} e^{-\frac{p^2}{\tau}} d\beta dp$ ", we can present the creation and annihilation on 1+1-de Sitter:

$$O_{e^{i\beta-p}} = \int_{\vec{x}\in S^1} \int_{\vec{x}.\vec{p}=0} e^{(i\beta-p)} |\Psi_a^\tau >< \Psi_a^\tau|\, \mu(d\beta, dp) = \sum_n e^{-\tau(n+\frac{1}{2})} \mid n+1 >< n \mid \quad (9)$$

$$O_{e^{-i\beta-p}} = \int_{\vec{x}\in S^1} \int_{\vec{x}.\vec{p}=0} e^{(-i\beta-p)} |\Psi_a^\tau >< \Psi_a^\tau|\, \mu(d\beta, dp) = \sum_n e^{-\tau(n-\frac{1}{2})} \mid n-1 >< n \mid \quad (10)$$

## 4. Conclusion

By using the coherent states and the Berezin quantization method, we showed that the creation and annihilation operators for a massive particle on 1+1-de Sitter space are respectively, the operators $O_{\bar{z}}$ and $O_z$ where $z = e^{-i\beta-p}$ (like simple harmonics oscillator).

## References


[1] A.Rabeie, E. Huguet, J. Renaud, *Wick ordering for coherent state quantization in 1 + 1 de Sitter space*, Physics Letters A 370, (2007), 123-125.

[2] B.Hall, J.J.Mitchell, *Coherent states on spheres*, J.Math.Phys. **43(3)**, (2002), 1211-1236.

[3] F.A.Berezin, *Quantization*, Math.USSR Izvestija 8, (1974), 1109-1165.

[4] J.P.Gazeau, *Coherent states in quantum physics*, WILEY-VCH Verlag GmbH and Co. KGaA, Weinheim, 2009.



A. RABEIE,
Department of Physics
Razi University of Kermanshah
Kermanshah
Iran
e-mail: rabeie@razi.ac.ir