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Second transpose of a dual valued derivation

M. Essmaili*

Abstract

Let A be a Banach algebra and X be an arbitrary Banach A-bimodule. In this paper, we study second transpose a derivation with value in dual Banach A-module X^*. Indeed, for a continuous derivation $D : A \longrightarrow X^*$ we obtain a necessary and sufficient condition such that the bounded linear map $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ to be a derivation, where Λ is composition of restriction and canonical injection maps.

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1. Introduction

The second transpose a derivation from a Banach algebra A into dual module A^* has been discussed in some papers, see for example [3], [4] and [5].

Dales, Rodriguez and Velasco in [3] studied second transpose a A^*-valued derivation $D : A \longrightarrow A^*$ and obtained conditions under which the second transpose $D'' : A^{**} \longrightarrow A^{***}$ is a derivation. Indeed, it is shown that D'' is a derivation if and only if $D''(A^{**}) \cdot A^{**} \subseteq A^*$ [3, Theorem 7.1]. Also in [5], the authors investigated the second transpose of a derivation D on Banach algebra A with value in X^*, where X is an arbitrary Banach A-bimodule. Indeed, they obtained a necessary and sufficient condition under which $D'' : A^{**} \longrightarrow X^{***}$ be a derivation and generalized some results in [3].

Weak amenability of second dual of Banach algebras has been studied with a different approach by Ghahramani, Loy and Willis in [4]. They considered some conditions under which the map $\Lambda \circ D'' : A^{**} \longrightarrow A^{***}$ to be a derivation, where Λ is composition of restriction and canonical injection maps as defined in Section 2. Indeed, they claimed that $\Lambda \circ D''$ is a derivation in the case where A is a left ideal of A or D is weakly compact, see also [2]. They used the fact that for each $F, G \in A^{**}$, the equality $\Lambda(D''(F) \cdot G) = (\Lambda \circ D''(F)) \cdot G$ holds on A^{**} if and only if it holds just on A.

Recently, the authors in [1] obtained a necessary and sufficient condition such that the map $\Lambda \circ D'' : A^{**} \longrightarrow A^{***}$ be a derivation as follows and showed that part of the proof of [4, Theorem 2.3] and [2, Lemma 4(ii)] have the same computational gap.

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Theorem 1.1. *([1, Theorem 3.1]) Let A be a Banach algebra and $D : A \longrightarrow A^*$ be a continuous derivation. Then the following are equivalent:*

(i) $\Lambda \circ D'' : A^{**} \longrightarrow A^{***}$ *is a derivation.*

(ii) *For every* $F, G \in A^{**}$, $\Lambda(D''(F) \cdot G) = (\Lambda \circ D''(F)) \cdot G$ *on A and*

$$\Lambda \circ D''(A^{**}) \subseteq WAP(A).$$

In this paper, our aim is to study of the map $\Lambda \circ D''$ in general case which is an extension of some results in [1]. Indeed, for an arbitrary Banach A-bimodule X and continuous derivation $D : A \longrightarrow X^*$, we obtain a necessary and sufficient condition under which the map $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ to be a derivation. As a consequence, in the case where D is a weakly compact derivation, we obtain a simple condition equivalent to the map $\Lambda \circ D''$ be a derivation.

2. Main results

Throughout the paper, X denotes a Banach space and X^* , X^{**} and X^{***} are first, second and third dual of X, respectively. The canonical embedding map $\kappa_X : X \longrightarrow X^{**}$ is defined by

$$\kappa_X(x)(f) = f(x) \qquad (x \in X, f \in X^*).$$

We also use the notation $\kappa_X(x) = \widehat{x}$. Furthermore, we consider the restriction map $R : X^{***} \longrightarrow X^*$ by the equation

$$R(\psi)(x) = \psi(\widehat{x}) \qquad (\psi \in X^{***}, x \in X).$$

Definition 2.1. *Let X be a Banach space. We define the bounded linear map $\Lambda : X^{***} \longrightarrow X^{***}$ by*

$$\Lambda(\psi) = \kappa_{X^*} \circ R(\psi) \qquad (\psi \in X^{***}).$$

Definition 2.2. *Let X, Y and Z be Banach spaces and $\pi : X \times Y \longrightarrow Z$ be a bounded bilinear map. The flip map $\pi^r : Y \times X \longrightarrow Z$ is regarded by the equation*

$$\pi^r(y, x) = \pi(x, y) \qquad (x \in X, y \in Y).$$

We also define the transpose of $\pi^* : Z^* \times X \longrightarrow Y^*$ by

$$\pi^*(z^*, x)(y) := z^*(\pi(x, y)) \quad (x \in X, y \in Y, z^* \in Z^*).$$

Let A be a Banach algebra and X be a Banach A-bimodule with the left module action $\pi_1 : A \times X \longrightarrow X$ and the right module action $\pi_2 : X \times A \longrightarrow X$. Similar to [5], we denote this module structure by the triple (π_1, X, π_2). It is straightforward to check that X^* is a Banach A-module where the left and right module actions are π_2^{r*r} and π_1^*, respectively. Moreover, X^{**} is a Banach $(A^{**}, \Box)$-module with the left and right module actions π_1^{***} and π_2^{***}. By this fact, we conclude that the triple $(\pi_2^{***r*r}, X^{***}, \pi_1^{****})$ is a Banach $(A^{**}, \Box)$-module.

Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. A bounded linear map $D : A \longrightarrow X$ is a derivation if

$$D(ab) = \pi_1(a, D(b)) + \pi_2(D(a), b) \qquad (a, b \in A).$$

Now, consider the A-module structure $(\pi_2^{r*r}, X^*, \pi_1^*)$ and suppose that $D : A \longrightarrow X^*$ is a derivation. In this section, we obtain a necessary and sufficient condition such that the map $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ be a derivation by regarding the A^{**}-module structure $(\pi_2^{***r*r}, X^{***}, \pi_1^{****})$.

Theorem 2.3. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. If $D : A \longrightarrow X^*$ is a continuous derivation, then for each $F, G \in A^{**}$,*

$$\Lambda \circ D''(F \Box G) = \Lambda(\pi_1^{****}(D''(F), G)) + \Lambda(\pi_2^{r*r***}(F, D''(G))).$$

We recall that in throughout this section, X^* is regarded as a A-module with the structure $(\pi_2^{r*r}, X^*, \pi_1^*)$ and moreover X^{***} is a A^{**}-module with the structure $(\pi_2^{***r*r}, X^{***}, \pi_1^{****})$.

The following theorem gives a necessary and sufficient condition such that the map $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ be a derivation.

Theorem 2.4. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. If $D : A \longrightarrow X^*$ is a continuous derivation, then the following are equivalent:*

(i) $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ is a derivation.

(ii) For each $F, G \in A^{**}$ we have

$$\Lambda(\pi_1^{****}(D''(F), G)) = \pi_1^{****}(\Lambda \circ D''(F), G).$$

Definition 2.5. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. We define set $\mathcal{W}$ by*

$$\mathcal{W} = \{f \in X^* : \text{the map } x^{**} \longrightarrow \langle \pi_1^{***}(G, x^{**}), f\rangle \text{ is weak}^* \text{ continuous on } X^{**} \text{, for each } G \in \mathcal{A}^{**}\},$$

In particular, let A be a Banach algebra $X = A$ and $\pi_1 = \pi_2 = m$, where m is the multiplication map A. it is easy to see that $\mathcal{W}$ is exactly the set of all almost periodic functionals on A.

By this fact, the following theorem may be regarded as a generalization of [1, Theorem 3.1].

Theorem 2.6. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. If $D : A \longrightarrow X^*$ is a continuous derivation, then the following are equivalent:*

(i) $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ is a derivation.

(ii) For each $F, G \in A^{**}$, the equality $\Lambda(\pi_1^{****}(D''(F), G)) = \pi_1^{****}(\Lambda \circ D''(F), G)$ holds on X and moreover

$$\Lambda \circ D''(A^{**}) \subseteq \mathcal{W}.$$

As a consequence, we have the following characterization for weakly compact derivations.

Corollary 2.7. *Let A be a Banach algebra, (π_1, X, π_2) be a Banach A-module and $D : A \longrightarrow X^*$ be a weakly compact derivation. Then $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ is a derivation if and only if $D''(A^{**}) \subseteq \mathcal{W}$.*

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M. ESSMAILI,
Department of Mathematics,
Kharazmi University,
Tehran, Iran.
e-mail: m.essmaili@khu.ac.ir