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Generalized SHift-Invariant Systems and Dual Frames for Subspaces

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Abstract

Let T_k denote translation by $k \in \mathbb{Z}^d$. Given countable collections of functions $\{\phi_j\}_{j\in J}$, $\{\tilde{\phi}_j\}_{j\in J} \subset L^2(\mathbb{R}^d)$ and assuming that $\{T_k\phi_j\}_{j\in J,k\in\mathbb{Z}^d}$ and $\{T_k\tilde{\phi}_j\}_{j\in J,k\in\mathbb{Z}^d}$ are Bessel sequences, we are interested in expansions

$$f = \sum_{j\in J}\sum_{k\in\mathbb{Z}^d} < f, T_k\tilde{\phi}_j > T_k\phi_j, \forall f \in \overline{span}\{T_k\phi_j\}_{k\in\mathbb{Z}^d}.$$

Our main result gives an equivalent condition for this to hold in a more general setting than described here, where translation by $k \in \mathbb{Z}^d$ is replaced by translation via the action of a matrix. As special cases of our result we find conditions for shift-invariant systems to generate a subspace frame with a corresponding dual having the same structure.In this paper we consider the important case of generalized shift-invariant systems and provide various ways of estimating the deviation from perfect reconstruction that occur when the systems do not form dual frames.The deviation from being dual frames will be measured either in terms of a perturbation condition or in terms of the deviation from equality in the duality conditions.

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1. Introduction

Given a real and invertible $d\times d$ matrix C, we define for $k \in F^d$ a generalized translation operator T_{Ck} acting on $f \in L^2(\mathbb{R}^d)$ by

$$(T_{Ck}f)(x) = f(x - Ck), x \in (\mathbb{R}^d)$$

A generalized shift-invariant system is a system [1–3] of the type $\{T_{C_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}^d}$ where $\{C_j\}_{j\in J}$ is a countable collection of real invertible $d \times d$ matrices, and $\{\phi_j\}_{j\in J} \subset L^2(\mathbb{R}^d)$. Generalized shift-invariant systems contain the classical wavelet systems and Gabor systems as special cases. Given the matrices $\{C_j\}_{j\in J}$, we are interested in functions $\{\phi_j\}_{j\in J}$, $\{\tilde{\phi}\}_{j\in J} \subset L^2(\mathbb{R}^d)$ for which $\{T_{C_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}^d}$ and $\{T_{C_jk}\tilde{phi}_j\}_{j\in J,k\in\mathbb{Z}^d}$ are Bessel sequences and the expansions

$$f = \sum_{j\in J}\sum_{k\in\mathbb{Z}^d} < f, T_{C_jk}\tilde{\phi}_j > T_{C_jk}\phi_j, \forall f \in \overline{span}\{T_{C_jk}\phi_j\}_{j\in J,k\in\mathbb{Z}^d}$$

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hold. If two sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ form a pair of dual frames for $\mathcal{H}$, each $f \in \mathcal{H}$ has a representation

$$f = \sum_{k=1}^{\infty} < f, g_k > f_k. \tag{1}$$

In signal processing terms this is expressed by saying that dual pairs of frames leads to perfect reconstruction.

2. Preliminaries and notations

we consider generalized shift-invariant systems of the type $\{T_{C_j k}\phi_j\}_{j\in J, k\in\mathbb{Z}^d}$, where $\{C_j\}_{j\in J}$ is a countable collection of real invertible $d\times d$ matrices, and $\{\phi_j\}_{j\in J} \subset L^2(\mathbb{R}^d)$. Letting C^T denote the transpose of an invertible matrix C, we use the notation

$$C^\sharp = (C^T)^{-1}.$$

For $f \in L^1(\mathbb{R}^d)$ we denote the Fourier transform by

$$\mathcal{F}f(\gamma) = \hat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i x.\gamma}dx,$$

where $x.\gamma$ denotes the inner product between x and γ . As usual, the Fourier transform [1] is extended to a unitary operator on $L^2(\mathbb{R}^d)$. Furthermore, With $E_b(x) := e^{2\pi i b.x}, b, x \in \mathbb{R}^d$, this yields the commutator relation

$$\mathcal{F}T_{Ck} = E_{-Ck}\mathcal{F}.$$

While we present the general theory for the d-dimensional case, our discussion of shift-invariant systems will take place in $L^2(\mathbb{R})$. Generalized shift-invariant systems in $L^2(\mathbb{R})$ will be denoted by

$$\{T_{a_j k}\phi_j\}_{j\in J, k\in\mathbb{Z}}, \quad \text{where} \quad a_j > 0, \phi_j \in L^2(\mathbb{R})$$

Most of our calculations rely on Fourier transformation techniques rather than general Hilbert space results.

Two frames $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames if (1) holds. Note that if $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences and (1) holds, then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are automatically frames. Given any Bessel sequence $\{f_k\}_{k=1}^{\infty}$ one can define a bounded operator $T : \ell^2(\mathbb{N}) \longrightarrow \mathcal{H}$ by $T\{c_k\}_{k=1}^{\infty} := \sum c_k f_k$; the operator T is called the synthesis operator or preframe operator [2]. It is easy to see that the adjoint operator is given by $T^* : \mathcal{H} \longrightarrow \ell^2(\mathbb{N}), T^*f = \{< f, f_k >\}_{k=1}^{\infty}$. Denoting the synthesis operators for two Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ by T, respectively, U, it is clear that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are dual frames if and only if $TU^* = I$.

We note that in the summation over $n \in \mathbb{Z}^d$, the indices j, n always appear in the combination $C_j^\sharp n$; let us consider all possible outcomes, i.e., let

$$\Lambda = \{C_j^\sharp n : j \in J, n \in \mathbb{Z}^d\}. \tag{2}$$

Given $\alpha \in \Lambda$, there might exist several pairs $(j, n) \in J \times \mathbb{Z}^d$ for which $\alpha = C_j^\sharp n$; let

$$J_\alpha = \{j \in J : \exists n \in \mathbb{Z}^d \text{such that } \alpha = C_j^\sharp n \quad (3)$$

Definition 2.1. *A generalized shift invariant system [3] in* $L^2(\mathbb{R})$ *is a collection of functions of the form* $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}, j\in J}$, *where* $\{\phi_j\}_{j\in J} \subset L^2(\mathbb{R})$ *and* $\{c_j\}_{j\in J}$ *is a countable collection of positive numbers.*

Definition 2.2. *Consider two GSI-systems* $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}, j\in J}$ *and* $\{T_{c_j k}\tilde{\phi}_j\}_{k\in\mathbb{Z}, j\in J}$

(i) *If*

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{1}{c_j} \int_{supp \hat{f}} |\hat{f}(\gamma + c_j^{-1} m)\hat{\phi}_j(\gamma)|^2 d\gamma < \infty$$

for all $f \in \mathcal{D}, (\mathcal{D} := \{f \in L^2(\mathbb{R}) \mid supp\hat{f} \text{is compact and} \hat{f} \in L^\infty(\mathbb{R})\})$, *we say that* $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}, j\in J}$ *satises the LIC condition.*

(ii) $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}, j\in J}$ *and* $\{T_{c_j k}\tilde{\phi}_j\}_{k\in\mathbb{Z}, j\in J}$ *satisfy the dual* α*-LIC condition if*

$$\sum_{j\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \frac{1}{c_j} \int_{supp \hat{f}} |\hat{f}(\gamma + c_j^{-1} m)\hat{\phi}_j(\gamma)\hat{\tilde{\phi}}_j(\gamma + c_j^{-1} m)| d\gamma < \infty$$

for all $f \in \mathcal{D}$. *We say that* $\{T_{c_j k}\phi_j\}_{j\in\mathbb{Z}}$ *satisfies the -LIC condition, if holds with* $\phi_j = \hat{\phi}_j$.

3. Main results

In this section we consider translates of a single function. The results obtained here will serve as starting point for the case of multiple generators.

Lemma 3.1. *Let* $\phi, \tilde{\phi} \in L^2(\mathbb{R}^d)$, *let* C *be a real and invertible* $d \times d$ *matrix, and assumethat* $\{T_{Ck}\phi\}_{k\in\mathbb{Z}^d}$ *and*$\{T_{Ck}\tilde{\phi}\}_{k\in\mathbb{Z}^d}$*are Bessel sequences. Then for all* $f \in L^2(\mathbb{R}^d)$,

$$\mathcal{F}(\sum_{k\in\mathbb{Z}^d} < f, T_{Ck}\tilde{\phi} > T_{Ck}\phi) = \frac{1}{|\det C|}\hat{\phi}(\gamma) \sum_{n\in\mathbb{Z}^d} \hat{f}(\gamma + C^\sharp n)\overline{\hat{\tilde{\phi}}(\gamma + C^\sharp n)}.$$

Our rst result below characterizes pairs of a frame $\{T_{C_j k}\phi_j\}_{k\in Z_d, j\in J}$ and corresponding generalized duals $\{T_{C_j k}\tilde{\phi}_j\}_{k\in Z_d, j\in J}$. The general classication is quite involved; the next sections show how it simplies in concrete cases.

Theorem 3.2. *Let* J *be a countable index set and consider sequences* $\{\phi_j\}_{j\in J}$ *and* $\{\tilde{\phi}_j\}_{j\in J}$ *in* $L^2(\mathbb{R}^d)$. *Let* $\{C_j\}_{j\in J}$ *be a sequence of invertible real matrices, and assume that* $\{T_{c_j k}\phi_j\}_{j\in\mathbb{Z}^d}$ *are Bessel sequences. Then the following are equivalent:*

(i) $f = \sum_{j\in J} \sum_{k\in\mathbb{Z}^d} < f, T_{c_j k}\tilde{\phi}_j > T_{c_j k}\phi_j, \forall f \in \overline{span}\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}^d, j\in J}$

(ii) *for all* $\ell \in J$, $m \in \mathbb{Z}^d$

$$\hat{\phi}_\ell(\gamma) = \sum_{j\in J} \frac{1}{|\det C_j|}\hat{\phi}_j(\gamma)(\sum_{n\in\mathbb{Z}^d} e^{-2\pi i C_\ell m . C_j^\sharp n})\hat{\phi}_\ell(\gamma + C_j^\sharp n)\overline{\hat{\tilde{\phi}}_\ell(\gamma + C_j^\sharp n)},$$

holds for a.e. γ. *If the conditions are satised, then* $\{T_{c_j k}\phi_j\}_{j\in\mathbb{Z}^d}$ *is a dual frame for* $\overline{span}\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}^d, j\in J}$.

Theorem 3.3. *Assume that the GSI-systems* $\{T_{c_j k}\phi_j\}_{k\in\mathbb{Z}, j\in J}$ *and* $\{T_{c_j k}\tilde{\phi}_j\}_{k\in\mathbb{Z}, j\in J}$ *are Bessel sequences and satisfy the dual* α*-LIC-condition for all* $f \in \mathcal{D}$*; denote the associated preframe operators by* T*, resp.,* U*. Then*

$$\|I - UT^*\| \leq \| \sum_{j\in J} \frac{1}{c_j} \overline{\hat{\phi}_j(\gamma)} \hat{\tilde{\phi}}_j(\gamma) - 1\|_\infty + \sum_{\alpha\in\Lambda\backslash\{0\}} \| \sum_{j\in J_\alpha} \overline{\hat{\phi}_j(\gamma)} \hat{\tilde{\phi}}_j(\gamma+\alpha)\|_\infty$$

for $\Lambda := \{C_j^{-1} n : j \in J, n \in \mathbb{Z}\}$ *and for* $\alpha \in \Lambda$*, let*

$$J_\alpha := \{j \in J : \exists n \in \mathbb{Z} such that \alpha = c_j^{-1} n\}.$$

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