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The relation between existence of admissible vectors and compactness of a locally compact group

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Abstract

Let G be a locally compact group and $\pi : G \longrightarrow U(H_\pi)$ be a unitary representation. In this article we will study on the existence of an admissible vector for irreducible representations and the compactness of G. In fact, it will be shown that if G is a compact group then every irreducible representation of G has an admissible vector, and also has a bounded cyclic vector. Conversely if G has property (T) and a finite irreducible representation of G has an admissible vector, then G is a compact group. Since containment of an irreducible representation π in the left regular representation λ_G is a necessary and sufficient condition for existence of admissible vector, hence it seems a natural object of weakly containment of these representation and therefore property (T) is peered.

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1. Introduction

There are interesting connections between admissible vectors and the compactness of the group G. Some results for containment of representations can be obtain by property (T). A locally compact group G has the property (T) if there exist a compact subset Q and a real number $\epsilon > 0$ such that, whenever π is a continuous unitary representation of G on a Hilbert space H for which there exists a vector $\eta \in H$ of norm 1 with $\sup_{x \in Q} \|\pi(x)\eta - \eta\| < \epsilon$, then there exists an invariant vector, namely a vector $\xi \neq 0$ in H such that $\pi(x)\xi = \xi$ for all $x \in G$. The vector η with above property is called $(Q, \epsilon)-$invariant. If for every compact subset Q and a real number $\epsilon > 0$ there exist a $(Q, \epsilon)-$invariant vector, it is written $1_G \prec \pi$ and if there exist invariant vectors, it is written $1_G < \pi$. If a group G has the property (T), then it is called a Kazhdan group. The trivial example of Kazhdan groups are compact groups. Compact groups can be characterised as the locally compact groups which are amenable and which have property (T). If a group G be compact, then every irreducible representation on G has the admissible vectors. In this article we have investigated conditions that confirm the converse statements. For example if a locally compact group G has a representation that has a invariant vector and a admissible vector, then G is compact.

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2. Definitions and Preliminaries

A unitary, strongly continuous representation, or simply a representation, of a locally compact group G is a group homomorphism $\pi : G \longrightarrow U(H_\pi)$, which is continuous when the right hand side is endowed with the strong operator topology, that is $x \longrightarrow \pi(x)\eta$ is continuous from G to H_π for each $\eta \in H_\pi$. Therefore, all coefficient functions of the type $G \ni x \longrightarrow \langle \xi, \pi(x)\eta \rangle \in \mathbb{C}$ are continuous, where $\xi, \eta \in H_\pi$. This is because the weak and strong operator topologies coincide on $U(H_\pi)$. See [4] for more detailes. The representation of G is called irreducible if H_π and $\{0\}$ are the only subspace of H_π that invariant under π. We use the notation $\pi_1 < \pi_2$ when π_1 is unitary equivalent with some subrepresentation of π_2, and we say that π_1 is contained in π_2. Let G be a locally compact group. The left regular representation λ_G acts on $L^2(G)$ by $(\lambda_G(x)f)(y) = f(x^{-1}y)$. Clearly this representation is unitary and for $f, g \in L^2(G)$, we have $(g * f^*)(x) = \langle g, \lambda_G(x)f \rangle$, where $f^*(x) = \overline{f(x^{-1})}$.
Let (π_1, H) and (ρ, K) be unitary representations of the topological group G. A representation π is called weakly contained in ρ, if every function of positive type associated to π can be approximated, uniformly on compact subsets of G, by finite sums of functions of positive type associated to ρ. This means for every ξ in H, every compact subset Q of G and every $\epsilon > 0$, there exist $\eta_1, ..., \eta_n$ in K such that, for all xQ,

$$| < \pi(x)\xi, \xi > - \sum_{i=1}^{n} < \rho(x)\eta_i, \eta_i > | < \epsilon.$$

We write for this $\pi \prec \rho$. By theorem 1.2.1 in [2], a topological group G has the property (T), if and only if " $1_G \prec \lambda_G$ then $1_G < \lambda_G$ ".

Let (π, H_π) denote a strongly continuous unitary representation of a locally compact group G. We endow G with it's left Haar measure. For each $\eta \in H_\pi$ associate the coeffitient operator $V_\eta : H_\pi \longrightarrow C_b(G)$ defined by $V_\eta\xi(x) = \langle \xi, \pi(x)\eta \rangle$. The vector $\eta \in H_\pi$ is called admissible if $V_\eta : H_\pi \longrightarrow L^2(G)$ is an isometery. In this case $V_\eta : H_\pi \longrightarrow L^2(G)$ is called the (generalized) **continuous wavelet transform**.

3. Main Results

We concentrated on existance of admissible vectors and in particular on solving this question: Is the compactness of a locally compact group G is equivalent to, "every irredusible representation of G has an admissible vector"?

The main theorem for existence of admissible vectors is the following theorem, Theorem 2.25 in [5].

Theorem 3.1. *Let π be an irreducible representation of G. π has admissible vectors iff $\pi < \lambda_G$.*

The following theorem that fined in[1], [2] and [3] shows the relation between amenability of a locally compact.

Theorem 3.2. *Let G be a locally compact group. The following are equivalent:*
(i) G is amenable;
(ii) Every irreducible, unitary representation of G is weakly contained in λ_G*;*
(iii) The trivial representation of G on $\mathbb{C}$ *(the one that maps every element of G to 1) is weakly contained in* λ_G*;*
(iv) λ_G *is amenable.*

Corollary 3.3. *Let G be a locally compact group. If every irreducible representation of G has an admissible vector then G is amenable.*

Proof. If π is an arbitrary irreducible representation which has admissible vectors, then by Theorem 3.1 $\pi < \lambda_G$. Therefore, π is weakly contained in λ_G and by Theorem 3.2, G is amenable. □

Remark 3.4. Note that the converse of 3.3 is not hold in general. In the other word amenability condition generally does not imply the existence of admissible vector for all irreducible representations. For an example for instance let $G = \mathbb{R}$ with the ordinary operation +,.. Since G is an abelian group, it is amenable. But λ_G has no irreducible subrepresentation, where λ_G is representation of $\mathbb{R}$ on $L^2(\mathbb{R})$ difined by $[\lambda_G(x)f](t) = f(t - x)$ (see [4] page 72 for more detailes).

It is seen that the condition "existence of admissible vector for all irreducible representation" is more stronger than amenability, and so we need a stronger condition such as compactness. In this context, the following theorem is a useful case.

Theorem 3.5. *Let G be a locally compact group. The following are equivalent:*
(i) G is compact;
(ii) 1_G *is contained in* λ_G*;*

Proof. See Theorem A.5.1 in [1] for detailes. □

Corollary 3.6. *Let G be a locally compact group. If there exists a representation of G such that has a non zero vector that is admissible and invariant, then G is compact.*

Proof. Let π is a representation of G such that has a non zero vector $\eta \in H_\pi$ that is admissible and invariant vector. then

$$\|\eta\|^2 = \|V_\eta\|^2 = \int_G | < \eta, \pi(x)\eta > |^2 d\mu(x) = \int_G | < \eta, \eta > |^2 d\mu(x) = \|\eta\|^2 \mu(G)$$

therefore μ is finite and then G is compact. □

Proposition 3.7. *Let G be a compact group. Then every irreducible representation of G has an admissible vector.*

Proof. It follows from Peter-weyl Theorem(iii) [[4], Theorem 5.12], and Theorem 3.1. □

Theorem 3.8. *Let G be a locally compact group. The following are equivalent:*
(i) G is compact;
(ii) G is amenable and has the property (T) .

Proof. See Theorem 1.1.6 in [1] for detailes. □

Proposition 3.9. *Let G be a compacrt group and let π and ρ be unitary representations of G. Then $\pi \prec \rho$ if and only if every irreducible subrepresentation of π is contained in ρ. Therefore if π is irreducible then $\pi \prec \rho$ is equivalent to $\pi < \rho$.*

Proof. See Theorem F.1.8 in [2] for detailes.

Proposition 3.10. *Let G be a locally compact group and has property (T). If there exist a finite irreduscible unitary representation of G that has an admissible vector, then G is amenable and therefore, by theorem 3.8 G is compact.*

Proof. If π be a finite irreduscible unitary representation of G that has a admissible vector then by 3.1, $\pi < \lambda_G$. Therfore by [2] Corollarry 5.9 λ_G is amenable, then by theorem 3.2, G is amenable and by theorem 3.5, G is compact. □

The following theorem explain the relation between existennce of admissible vectors and compactness in particular case. See [5], Theorem 2.35.

Theorem 3.11. *Let G be a SIN-group, i.e., every neighborhood of unity contains a conjugation-invariant neighborhood. If G has a discrete series representation, i.e., there exists irreducible representation that has admissible vectors then G is compact. In particular, if G is discrete and has a discrete series representation, then G is finite. If G is abelian and has a discrete series representation, then G is compact.*

References

[1] B. Bekka, Amenable unitary representations of locally compact groups, *Invent. math*, **100**, (1990) 383-401.

[2] B. Bekka, P.Harpe and A.Valette, Kazhdan's property (T), Cambridge University Press, New York, 2008.

[3] A. Hulanicki, Groups whose regular representation weakly contains all unitary representations , *Studia Math* **24** (1964) 37-59.

[4] G.B. Folland, A course in abstract harmonic analysis, CRC Press, Boca Raton, 1995.

[5] H. Fuhr, Abstract Harmonic Analysis of Continuous Wavelet Transforms, Springer-Verlag Berlin Heidelberg, 2005.

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